$$\frac{z}{z+3} = \frac{\bar{z}}{\bar{z}+\alpha i}$$

Per quali $\alpha \in \mathbb{R}$ l'eq^{ne} ammette anche soluzioni non nulle.

$\boxed{z \neq -3}$ $\quad \bar{z} \neq -\alpha i$

$\Updownarrow$

$\boxed{z \neq \alpha i}$

$$(\bar{z}+\alpha i) z = (z+3)\bar{z}$$

$$\cancel{\bar{z}z} + \alpha i z = \cancel{z\bar{z}} + 3\bar{z}$$

$$\alpha i z = 3\bar{z} \qquad z = x+iy \quad x,y \in \mathbb{R}.$$

$$\alpha i (x+iy) = 3(x-iy).$$

$$i\alpha x - \alpha y = 3x - 3iy.$$

$$\begin{cases} 3x + \alpha y = 0 \\ \alpha x + 3y = 0 \end{cases}$$

se $9-\alpha^2 \neq 0$, cioè $\alpha \neq \pm 3$, solo la sol^{ne} nulla.

Se $\alpha = 3$, viene $\begin{cases} x+y=0 \\ x+y=0 \end{cases} \iff y = -x.$

se $\boxed{\alpha = 3}$ tutti i complessi della forma $\boxed{z = x(1-i) \quad x \in \mathbb{R}}$ sono soluzioni.

$\boxed{\alpha = 3 \quad \text{OK}}$

$\boxed{\alpha = -3}$ $\quad \begin{cases} x-y=0 \\ -x+y=0 \end{cases}$ $\quad \boxed{y = x}$

$\boxed{z = x(1+i)}$ $\quad \forall x \in \mathbb{R}$

$\boxed{\text{Risposta: } \alpha = \pm 3.}$

$$\int \frac{\sqrt{x}+1}{\sqrt[3]{x}-1}\,dx \qquad \sqrt{x} = \left(\sqrt[6]{x}\right)^3, \quad \sqrt[3]{x} = \left(\sqrt[6]{x}\right)^2$$

sost. $\sqrt{x} = t$ $\qquad \boxed{x = t^2 \quad dx = 2t\,dt}$

$\sqrt[3]{x} = t^{2/3}$

$$= 2\int \frac{t+1}{t^{2/3}-1}\, t\,dt = \qquad \boxed{t^{1/3} = s} \qquad t = s^3 \quad dt = 3s^2 ds$$

$\Rightarrow s = \sqrt[3]{t} = \sqrt[3]{\sqrt{x}} = \sqrt[6]{x}$

$$= 2\int \frac{(s^3+1)\,s^3}{s^2-1}\, 3s^2\,ds = 6\int \frac{s^8+s^5}{s^2-1}\,ds \quad \leftarrow$$

Si poteva porre dall'inizio $\sqrt[6]{x} = t \Rightarrow \sqrt[3]{x} = t^2,\ \sqrt{x} = t^3$

$x = t^6 \qquad dx = 6t^5\,dt$

$$\int \frac{\sqrt{x}+1}{\sqrt[3]{x}-1}\,dx = \int \boxed{\frac{t^3+1}{t^2-1}}\, 6t^5\,dt \quad \leftarrow$$

$$\boxed{\frac{t^3+1}{t^2-1}} = \frac{\cancel{(t+1)}(t^2-t+1)}{\cancel{(t+1)}(t-1)}$$

In definitiva, viene

$$\int \boxed{\frac{t^3+1}{t^2-1}}\, 6t^5\,dt = 6\int \frac{t^7-t^6+t^5}{t-1}\,dt$$

$$\frac{t^3+1}{t^2-1} = \frac{\cancel{(t+1)}(t^2-t+1)}{\cancel{(t+1)}(t-1)}$$

e poi si fa la divisione.

$$\int \frac{\sin(2x)-\cos x}{\sin x + 9\sin^3 x}\,dx = \int \frac{2\sin x\cos x - \cos x}{\sin x + 9\sin^3 x}\,dx =$$

$$= \int \frac{(2\sin x - 1)\cos x}{\sin x(1+9\sin^2 x)}\,dx \qquad t=\sin x \qquad dt = \cos x\,dx$$

$$= \int \frac{2t-1}{t(1+9t^2)}\,dt = \int \left(\frac{A}{t} + \frac{18Bt+C}{9t^2+1}\right)dt =$$

$$2t-1 = A(9t^2+1) + 18Bt^2 + Ct$$

$$\begin{cases} 0 = 9A+18B \\ 2 = C \\ -1 = A \end{cases} \qquad \boxed{C=2} \quad \boxed{A=-1} \quad \boxed{B = -\frac{A}{2} = \frac{1}{2}}$$

$$= -\log|t| + \frac{1}{2}\log(9t^2+1) + 2\int \frac{dt}{9t^2+1}$$

$$= \quad '' \qquad '' \quad + \frac{2}{3}\arctan(3t)\Big|_{t=\sin x}$$

$$\frac{z^2\,|z|}{|z|-8} = 4iz \qquad \boxed{z=0} \text{ è sol}^{\text{ne}}$$

divido per z.

$$\frac{z\,|\bar z|}{|z|-8} = 4i \qquad z = x+iy \qquad \boxed{x,y\in\mathbb{R}}$$

$$|z| = |\bar z| = \sqrt{x^2+y^2}$$

$$(x+iy)\sqrt{x^2+y^2} = 4i\left(\sqrt{x^2+y^2}-8\right) \qquad \boxed{|z|\neq 8}$$

$$\begin{cases} x\sqrt{x^2+y^2} = 0 \\ y\sqrt{x^2+y^2} = 4\left(\sqrt{x^2+y^2}-8\right)\end{cases}$$

$\sqrt{x^2+y^2}=0 \iff z=0$ già trovata

$x=0$

$$y\,|y| = 4(|y|-8)$$

$$\begin{cases} y>0 \\ y^2 = 4y-32 \end{cases}$$

$$y^2-4y+32=0$$

nessuna solne

$$\begin{cases} y<0 \\ -y^2 = -4y-32 \end{cases}$$

$$y^2-4y-32=0$$

$$y = 2\pm\sqrt{4+32} = 2\pm 6 = \begin{cases}\boxed{-4} \\ 8\end{cases}$$

Sol$^{\text{ni}}$ $\boxed{z=0}$ e $\boxed{z=-4i}$

Si poteva risolvere anche in forma polare:

$$z\,|\bar z| = 4i\left(|z|-8\right) \qquad z = |z|\,e^{i\varphi}$$

$$|z|^2 e^{i\varphi} = 4i\left(|z|-8\right)$$

1° caso $\quad |z|>8 \Rightarrow |z|^2 e^{i\varphi} = 4\underbrace{(|z|-8)}_{>0}\, e^{i\frac{\pi}{2}}$

$$\begin{cases} |z|^2 = 4(|z|-8) \\ \varphi = \frac{\pi}{2}+2k\pi \end{cases}$$

$$|z|^2-4|z|+32=0$$

nessuna sol$^{\text{ne}}$ reale.

2° caso $|z|<8 \Rightarrow |z|^2 e^{i\varphi} = 4(8-|z|)(-i) =$

$= 4(8-|z|)e^{-i\frac{\pi}{2}}$.

$$\begin{cases} |z|^2 = 32-4|z| \\ \varphi = -\frac{\pi}{2} (+2k\pi) \end{cases}$$

$|z|^2+4|z|-32=0$

$|z| = -2 \pm \sqrt{4+32} =$

$= -2 \pm 6 = \begin{cases} -8 & \text{non acc.} \\ 4 & \text{ok.} \end{cases}$

$|z|=4$

$\varphi = -\frac{\pi}{2}$

$$\boxed{z = 4e^{-i\pi/2} = -4i}$$

$$\int x^3 \sin(x^2+5)\,dx = \int x^2 \cdot x \sin(x^2+5)\,dx =$$

$x^2+5=t \qquad 2x\,dx = dt \qquad x^2 = t-5$

$$= \frac{1}{2}\int (t-5)\sin t\,dt =$$

$$\left[\begin{array}{ll} f'(t) = \sin t, & f(t) = -\cos t \\ g(t) = t-5 & g'(t) = 1 \end{array}\right]$$

$$= \frac{1}{2}\left[-(t-5)\cos t + \int \cos t\,dt\right] =$$

$$= \frac{1}{2}\left[(5-t)\cos t + \sin t\right] + c$$

$$\int_{-1}^{1} \log\left(1+\sqrt[3]{|x|}\right)dx = \boxed{2\int_0^1 \log\left(1+\sqrt[3]{x}\right)dx}$$

$$\left[\begin{array}{lll} \sqrt[3]{x} = t & x = t^3 & dx = 3t^2\,dt \\ & x=0 \Rightarrow t=0 & x=1 \Rightarrow t=1 \end{array}\right]$$

$$= 2\int_0^1 3t^2 \log(1+t)\,dt =$$

$$\left[\begin{array}{ll} f'(t)=3t^2 & f(t)=t^3 \\ g(t)=\log(1+t) & g'(t)=\dfrac{1}{1+t} \end{array}\right]$$

$$= 2\left[\underbrace{t^3\log(1+t)\Big|_0^1}_{\log 2} - \underbrace{\int_0^1 \frac{t^3}{1+t}\,dt}_{\parallel}\right]$$

$1+t=s$

$t=s-1$

$dt=ds$

$t=0 \Rightarrow s=1$

$t=1 \Rightarrow s=2$

$$-\int_1^2 \frac{(s-1)^3}{s}\,ds =$$

$$= -\int_1^2 \frac{s^3-3s^2+3s-1}{s}\,ds =$$

$$= -\int_1^2 \left(s^2-3s+3-\frac{1}{s}\right)ds =$$

$$= \left(-\frac{s^3}{3}+\frac{3}{2}s^2-3s+\log|s|\right)\Big|_1^2 =$$

$$= -\frac{7}{3}+\frac{9}{2}-3+\log 2$$

$$2\int_0^1 1\cdot\log\left(1+\sqrt[3]{x}\right)dx = 2\left[x\log\left(1+\sqrt[3]{x}\right)\Big|_0^1 - \frac{1}{3}\int_0^1 \frac{\overset{\sqrt[3]{x}}{\cancel{x}}}{1+\sqrt[3]{x}}\,\frac{1}{\cancel{x^{2/3}}}\,dx\right]$$

$$= 2\left[\log 2 - \frac{1}{3}\int_0^1 \frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}\,dx\right]$$

sost $\sqrt[3]{x}=t \quad x=t^3 \quad dx=3t^2\,dt$

Area di $E = \left\{(x,y)\in\mathbb{R}^2 : x \geq 0,\ \underbrace{\frac{e^{8x}}{4e^{4x}-3}}_{\text{sempre} > 0 \text{ per } x\geq 0} \leq y \leq 1\right\}$

$$\frac{e^{8x}}{4e^{4x}-3} \leq 1 \iff e^{8x} \leq 4e^{4x}-3 \iff$$

$$e^{8x} - 4e^{4x} + 3 \leq 0$$

$$t = e^{4x} \quad t^2 - 4t + 3 \leq 0$$

$$1 \leq t \leq 3$$

$$1 \leq e^{4x} \leq 3$$

$$0 \leq 4x \leq \log 3$$

$$\boxed{0 \leq x \leq \frac{\log 3}{4}}$$

$$E = \left\{(x,y) : 0 \leq x \leq \frac{\log 3}{4},\ \frac{e^{8x}}{4e^{4x}-3} \leq y \leq 1\right\}.$$

$$\text{Area } E = \int_0^{\frac{1}{4}\log 3}\left(1 - \frac{e^{8x}}{4e^{4x}-3}\right)dx =$$

$$= \frac{1}{4}\log 3 - \int_0^{\frac{1}{4}\log 3} \frac{e^{4x}\,\boxed{e^{4x}\,dx}}{4e^{4x}-3}$$

$$t = e^{4x} \qquad dt = 4e^{4x}\,dx$$

$$x = 0 \Rightarrow t = 1$$

$$x = \frac{1}{4}\log 3 \Rightarrow t = e^{\log 3} = 3$$

$$= \frac{1}{4}\log 3 - \frac{1}{4}\int_1^3 \frac{t - \frac{3}{4} + \frac{3}{4}}{4t-3}\,dt =$$

$$= \frac{1}{4}\log 3 - \frac{1}{16}\cdot 2 - \frac{3}{16}\int_1^3 \frac{dt}{4t-3} =$$

$$= \quad '' \quad -\frac{1}{8} - \frac{3}{16}\cdot\frac{1}{4}\log(4t-3)\Big|_1^3 =$$

$$= \frac{1}{4}\log 3 - \frac{1}{8} - \underbrace{\frac{3}{64}\left(\log 9\right)}_{-\frac{3}{32}\log 3.}$$

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1) $\int x\cos(4x-3)\,dx$

2) Formula iterativa per $I_n(x) = \int x^{2n+1}\cos(4x-3)\,dx$

$$\int x\cos(4x-3)\,dx =$$

$$\left[\begin{array}{ll} f'(x) = \cos(4x-3) & f(x) = \frac{1}{4}\sin(4x-3) \\ g(x) = x & g'(x) = 1 \end{array}\right]$$

$$= \frac{1}{4}x\sin(4x-3) - \frac{1}{4}\int \sin(4x-3)\,dx$$

$$= \quad '' \quad + \frac{1}{16}\cos(4x-3) + c.$$

$$\underline{I_n(x)} = \int \underline{x^{2n+1}}\cos(4x-3)\,dx =$$

$$\left[\begin{array}{ll} f'(x) = \cos(4x-3) \Rightarrow & f(x) = \frac{1}{4}\sin(4x-3) \\ g(x) = x^{2n+1} \Rightarrow & g'(x) = (2n+1)x^{2n} \end{array}\right]$$

$$= \frac{1}{4}x^{2n+1}\sin(4x-3) - \frac{2n+1}{4}\int x^{2n}\sin(4x-3)\,dx$$

$$\left[\begin{array}{ll} f'(x) = \sin(4x-3) & f(x) = -\frac{1}{4}\cos(4x-3) \\ g(x) = x^{2n} & g'(x) = 2n\,x^{2n-1} \end{array}\right]$$

$$= \frac{1}{4} x^{2n+1} \sin(4x-3) - \frac{2n+1}{4}\left[-\frac{1}{4} x^{2n} \cos(4x-3) + \frac{2n}{4} \underbrace{\int x^{2n-1} \cos(4x-3)\, dx}_{I_{n-1}(x)}\right]$$

In definitiva

$$\boxed{I_n(x) = \frac{1}{4} x^{2n+1} \sin(4x-3) + \frac{(2n+1)}{16} x^{2n} \cos(4x-3) + - \frac{(2n+1)n}{8} I_{n-1}(x)}$$