Handout: Difference equations

Giovanni Di Bartolomeo

1 Powers and series

Recall that

$$a^0 = 1 \tag{1}$$

$$a^\infty = \begin{cases} 0 & if \ \ |a| < 1 \\ sign(a) \cdot \infty & if \ \ |a| > 1 \end{cases} \tag{2}$$

It is trivial to show that $1^\infty = 1$, whereas $(-1)^\infty$ is not defined.

Recall that (**geometric series**)

$$\sum\nolimits_{i=0}^{n-1} a^i = 1 + a + a^2 + a^3 + .. + a^{n-1} = \frac{1-a^n}{1-a} \tag{3}$$

$$\sum\nolimits_{i=0}^{\infty} a^i = \begin{cases} \frac{1}{1-a} & if \ \ |a| < 1 \\ sign(a) \cdot \infty & otherwise \end{cases} \tag{4}$$

Note that $\sum\nolimits_{i=0}^{n-1} a^i = \sum\nolimits_{i=1}^{n} a^{i-1}$.

2 Two dynamic equations

Capital accumulation law. Existing capital depreciates over time at a fixed rate δ. The capital stock in the beginning of next period is given by the non-depreciated part of current-period capital, δK_{t-1}, plus contemporaneous investment, I_t. Note $\delta \in (0, 1)$.

$$K_t = \delta K_{t-1} + I_t \tag{5}$$

It's a backward–looking equation. History matters:

$$\cdots\cdots \ (t-1) \triangleright \triangleright \triangleright \triangleright \triangleright \ (t) \triangleright \triangleright \triangleright \triangleright \triangleright \ (t+1) \ \cdots\cdots \blacktriangleright$$

Past values affect current values.

Asset pricing equation. The current price, A_t, is equal to the discounted value of expected price, $E_t A_{t+1}$, plus the dividend, D_t. Note $\beta \in (0,1)$.

$$A_t = \beta \left(E_t A_{t+1} + D_t \right) \tag{6}$$

It's a forward–looking equation. Expectations matter:

$$\cdots\cdots \; (t-1) \triangleleft \triangleleft \triangleleft \triangleleft \triangleleft (t) \triangleleft \triangleleft \triangleleft \triangleleft \triangleleft (t+1) \; \cdots\cdots \blacktriangleright$$

Future values affect current values.

Now, we look at the steady states. Imposing the steady state (i.e., for $t \longrightarrow \infty$, $K_t = K_{t+1} = K$, $A_t = E_t A_{t+1} = A$, $I_t = I$, $D_t = D$), capital accumulation law and asset pricing equation implies:

$$K = \delta K + I \Longrightarrow K = \frac{I}{1-\delta} \tag{7}$$

$$A = \beta A + \beta D \Longrightarrow A = \frac{\beta D}{1-\beta} \tag{8}$$

If capital accumulation and asset pricing converge, they should converge to K and A.

2.1 Capital accumulation: Backward solution

Assume fixed investment $I_t = I$, at any t, it should be true that

$$K_t = \delta K_{t-1} + I \tag{9}$$

Note that $K_{t-1} = \delta K_{t-2} + I$, therefore

$$K_t = \delta \left(\delta K_{t-2} + I \right) + I = \delta^2 K_{t-2} + (1+\delta) I \tag{10}$$

But $K_{t-2} = \delta K_{t-3} + I$, hence

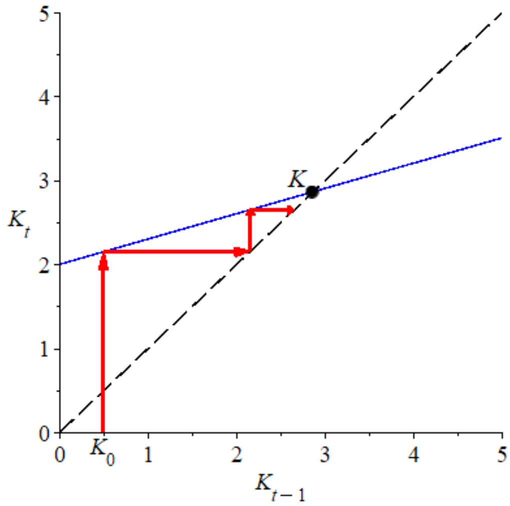
$$K_t = \delta^2 \left(\delta K_{t-3} + I \right) + (1+\delta) I = \delta^3 K_{t-3} + \left(1 + \delta + \delta^2 \right) I \tag{11}$$

Going back to K_0 (t periods in the past), we get

$$K_t = \delta^t K_0 + \left(1 + \delta + \delta^2 ... + \delta^{t-1} \right) I = \delta^t K_0 + \frac{1-\delta^t}{1-\delta} I \tag{12}$$

which converges to $K = I\left(1-\delta\right)^{-1}$ as $t \longrightarrow \infty$ when $|\delta| < 1$. Otherwise, it explodes.

Consider $K_t = 0.5K_{t-1} + 2$, starting from K_0, we can build



Starting from K_0, K_t evolves according to

$$K_t = \delta K_{t-1} + I \Longrightarrow K_1 = \delta K_0 + I \tag{13}$$

It follows

$$K_2 = \delta K_1 + I = \delta\left(\delta K_0 + I\right) + I = \delta^2 K_0 + \left(1+\delta\right) I \tag{14}$$

$$K_3 = \delta K_2 + I = \delta^3 K_0 + \left(1+\delta+\delta^2\right) I ... \tag{15}$$

and at n

$$...K_n = \delta^n K_0 + I\sum_{i=1}^{n} \delta^{i-1} = \delta^n K_0 + \frac{1-\delta^n}{1-\delta} I... \tag{16}$$

Finally,

$$K_\infty = \delta^\infty K_0 + I\sum_{i=1}^{\infty} \delta^{i-1} \tag{17}$$

If $|\delta| < 1$, then $\delta^\infty = 0$ and $\sum_{i=1}^{\infty} \delta^{i-1} = \frac{1}{1-\delta}$. It follows

$$K_\infty = \frac{I}{1-\delta} = K \tag{18}$$

2.2 Asset pricing equation: Forward solution

Abstracting from expectations, the equation

$$A_t = \beta \left(E_t A_{t+1} + D \right) \tag{19}$$

can be written as

$$A_{t+1} = \frac{1}{\beta} A_t + D \tag{20}$$

The above equation should also hold at t, so

$$A_t = \frac{1}{\beta} A_{t-1} + D \tag{21}$$

Now try to solve it as before, we should note that $\beta \in (0, 1)$, therefore $\left|\beta^{-1}\right| > 1$ and A_t does not converge to the steady state, but it explodes!!!

Indeed, there are two ways to solve a linear-difference equation: Iterating backward or iterating forward. The backward-looking dynamics stem, e.g., from identities linking today's capital stock with last period's capital stock and this period's investment, i.e., $K_t = (1 - \delta)K_{t-1} + I_t$. The forward-looking dynamics stem from optimizing behavior: What agents expect to happen tomorrow is very important for what they decide to do today.

Almost all economic transactions rely crucially on the fact that the economy is not a "one-period game." Economic decisions have an intertemporal element to them. A key issue in macroeconomics is how people formulate expectations about the in the presence of uncertainty. Modelling this idea requires an assumption about how people formulate expectations.

The DSGE approach relies on the idea that people have so-called rational expectations, introduced by Robert Lucas and Thomas Sargent in the 1970s. Rational Expectations usually means:

- Agents use publicly available information in an efficient manner. Thus, they do not make systematic mistakes when formulating expectations.

- Agents understand the structure of the model economy and base their expectations of variables on this knowledge.

Moreover, it is not rational for them to expect to have a different expectation next period for the expectations on y_{t+2} than the one that I have today, i.e.,

$$E_t \left(E_{t+1} y_{t+2} \right) = E_t y_{t+2}$$

This is known as the **Law of Iterated Expectations** (LIE). The point is that, today (Monday), my prediction about the wether Wednesday is the same prediction about the prediction I will do Tuesday. Probably, my prediction Tuesday will be different because I will have more information, but today I have not more information!

REs is clearly a strong assumption. The structure of the economy is complex and in truth nobody truly knows how everything works. But one reason for using REs as a baseline assumption is that once one has specified a particular model of the economy, any other assumption about expectations means that people are making systematic errors, which is inconsistent with rationality. Still, behavioral economists have now found lots of examples of deviations from rationality in people's economic behavior. But REs requires one to be explicit about the full limitations of people's knowledge and exactly what kind of mistakes they make. And while REs is a clear baseline, once one moves away from it there are lots of essentially *ad hoc* potential alternatives. Like all models, REs models need to be assessed on the basis of their ability to fit the data.

Consider again the equation $A_t = \beta E_t A_{t+1} + \beta D_t$. Note that $A_{t+1} = \beta E_{t+1} A_{t+2} + \beta E_{t+1} D_{t+2}$, therefore

$$\begin{aligned} A_t &= \beta E_t \left[\beta E_{t+1} A_{t+2} + \beta E_{t+1} D_{t+1}\right] + \beta D_t = \\ &= \beta^2 E_t A_{t+2} + \beta^2 E_t D_{t+1} + \beta D_t \end{aligned} \tag{22}$$

where we used: $E_t E_{t+1} X_t = E_t X_t$ (LIE).

But $A_{t+2} = \beta E_{t+2} A_{t+3} + \beta D_{t+2}$, taking the expectations and using LIE, we obtain

$$\begin{aligned} A_t &= \beta^3 E_t \left[E_{t+2} A_{t+3} + E_{t+2} D_{t+2}\right] + \beta^2 E_t D_{t+1} + \beta D_t = \\ &= \beta^3 E_t A_{t+3} + \beta^3 E_t D_{t+2} + \beta^2 E_t D_{t+1} + \beta D_t \end{aligned} \tag{23}$$

Iterating until n, we get

$$\begin{aligned} A_t &= \beta^n E_t A_{t+n} + \beta^n E_t D_{t+n-1} + \beta^{n-1} E_t D_{t+n-2} + \ldots + \beta D_t \\ &= \beta^n E_t A_{t+n} + \sum\nolimits_{i=1}^{n} \beta^i E_t D_{t+i-1} \end{aligned} \tag{24}$$

Assuming that the expectations converge to the steady state value ($E_t A_\infty = A$), we obtain that

$$\lim_{n \longrightarrow \infty} \beta^n E_t (A_{t+n} - A) = 0 \tag{25}$$

when $|\beta| < 1$; otherwise it explodes.

The limit amounts to a statement that A_t can't grow too fast.

Then for $|\beta| < 1$, we have

$$A_t = \sum\nolimits_{i=0}^{\infty} \beta^{i+1} E_t D_{t+i} \tag{26}$$

Asset prices should equal a discounted present-value sum of expected future dividends (dividend-discount model.)

It can be also check that if D_{t+i-1} is constant, $D_t = D$, then

$$A_\infty = A = \beta D (1 - \beta)^{-1} \tag{27}$$

3 Linear difference equation systems

3.1 A simple system

Consider the following linear system:

$$\begin{bmatrix} K_t \\ E_t A_{t+1} \end{bmatrix} = \begin{bmatrix} \delta & 0 \\ 0 & \beta^{-1} \end{bmatrix} \begin{bmatrix} K_{t-1} \\ A_t \end{bmatrix} + \begin{bmatrix} I \\ D \end{bmatrix} + \begin{bmatrix} \varepsilon_t^I \\ \varepsilon_t^D \end{bmatrix} \tag{28}$$

where ε_t^I and ε_t^D are two white noise shocks that can temporary deviate the system from the steady state but do not affect its stability.

Defining $X_{t+1} = [K_t, A_{t+1}]'$, $Y = [I, D]'$, $E_t = [\varepsilon_t^I, \varepsilon_t^D]'$, in a more compact form, it becomes

$$E_t X_{t+1} = A X_t + Y + E_t \tag{29}$$

Stability? The system summarizes the same equations studied before. Therefore, it requires $|\delta| < 1$ and $|\beta| < 1 \implies \left|\beta^{-1}\right| > 1$. Note that δ and β^{-1} are the eigenvectors of A.

3.2 The general case

Any DSGE is a linear dynamic model that can be written in a generalized state-space form:

$$\Upsilon E_t X_{t+1} = \Gamma X_t + \Phi V_{t+1} \tag{30}$$

where X_t is a vector of stationary variables and V_t is a vector of structural shocks. Υ, Γ, and Φ are coefficient matrices.

Models of this form (generalised linear rational expectations models) can be solved relatively easily by computer. Many techniques available to solve this class of models. We use Blanchard-Kahn.

By taking the inverse matrix of Υ, we transform the state-space form as follows (reduced state-space form):

$$E_t X_{t+1} = \Psi X_t + \Xi V_{t+1}$$

where $\Psi = \Upsilon^{-1}\Gamma$ and $\Xi = \Upsilon^{-1}\Phi$.

The model can be partitioned as:

$$X_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} \tag{31}$$

where x_{1t} are the backward-looking predetermined variables, x_{2t} are the forward-looking control variables.

The partitioned state-space form is:

$$\begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = \Psi \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \Xi V_{t+1} \tag{32}$$

By using the Jordan decomposition, matrix Ψ can be expressed as:

$$\Psi = \Lambda^{-1} J \Lambda \tag{33}$$

where J is a diagonal matrix consisting whose diagonal elements are the eigenvalues of Ψ. The eigenvalues are ordered in increasing absolute value in moving from left to right.

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \tag{34}$$

where the eigenvalues in J_1 lie inside the unit circle (stable roots) and those in J_2 lie outside the unit circle (unstable roots).

Λ is the matrix of the eigenvectors and is partitioned as

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \tag{35}$$

If the number of explosive eigenvalues is equal to the number of nonpredetermined variables, the system is stable and a unique solution exists (determinacy).

Formally,

$$\begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = \Lambda^{-1} J \Lambda \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \Xi V_{t+1} \tag{36}$$

Premultiply the system by Λ, we get

$$\begin{bmatrix} \widetilde{x}_{1t+1} \\ E_t\widetilde{x}_{2t+1} \end{bmatrix} = J \begin{bmatrix} \widetilde{x}_{1t} \\ \widetilde{x}_{2t} \end{bmatrix} + \Theta V_{t+1} \tag{37}$$

where

$$\begin{cases} \begin{bmatrix} \widetilde{x}_{1t+1} \\ E_t\widetilde{x}_{2t+1} \end{bmatrix} = \Lambda \begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} \\ \Theta = \Lambda\Xi \end{cases} \tag{38}$$

This transformation de-couples the system $\Longrightarrow$ nonpredetermined variables depend on the explosive eigenvalues of Ψ, contained in J_2.

Proposition 1 (Blanchard-Kahn conditions) *The solution of the rational expectations model is unique if the number of unstable eigenvalues of the system is exactly equal to the number of forward-looking (control) variables.*

Matrix J must contain a number of eigenvalues greater than 1 in magnitude equal to the number of forward-looking variables. The system is determined and has saddle-path stability, i.e., we have a unique stationary solution.

Other cases: If the number of explosive eigenvalues is smaller (greater) than the number of forward variables the system is undetermined (has no solutions). Undetermined system means that there are infinite solutions.

Remark 2 (A useful property) *The # of eigenvalues greater than one in magnitude (unstable roots) in J is the same of the # of eigenvalues smaller than one in magnitude (stable roots) in J^{-1}.*

It is useful as it is often easier to check the stable roots.

4 An example: A simple instrumental rule

Let us consider the following model:

$$\begin{cases} (1)\ \ \widetilde{y}_t = E_t\widetilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - r_t^n\right) \\ (2)\ \ \pi_t = \beta E_t\pi_{t+1} + k\widetilde{y}_t \\ (3)\ \ i_t = \phi_\pi \pi_t + \phi_y \widetilde{y}_t \end{cases} \tag{39}$$

where the natural real interest rate evolves according to $r_t^n = \rho_r r_{t-1}^n + \varepsilon_t^r$ with $\varepsilon_t^r \backsim N(0,1)$. Note: (1) IS curve; (2) New Keynesian Phillips Curve (NKPC); and (3) Taylor rule.

Plugging the Taylor rule and the NKPC into the IS yields

$$\begin{cases} \widetilde{y}_t = E_t\widetilde{y}_{t+1} - \frac{1}{\sigma}\left[\phi_\pi\left(\beta E_t\pi_{t+1} + k\widetilde{y}_t\right) + \phi_y\widetilde{y}_t - E_t\pi_{t+1} - r_t^n\right] \\ \pi_t = \beta E_t\pi_{t+1} + k\widetilde{y}_t \end{cases} \tag{40}$$

$$\begin{cases} \widetilde{y}_t + \frac{\phi_y}{\sigma}\widetilde{y}_t + \frac{\phi_\pi k}{\sigma}\widetilde{y}_t = E_t\widetilde{y}_{t+1} + \frac{1}{\sigma}(1-\phi_\pi\beta)E_t\pi_{t+1} + \frac{1}{\sigma}r_t^n \\ \pi_t = \beta E_t\pi_{t+1} + k\widetilde{y}_t \end{cases}$$

$$\begin{cases} \widetilde{y}_t = \frac{1}{\sigma+\phi_y+\phi_\pi k}\left[\sigma E_t\widetilde{y}_{t+1} + (1-\phi_\pi\beta)E_t\pi_{t+1} + r_t^n\right] \\ \pi_t = \beta E_t\pi_{t+1} + \frac{k}{\sigma+\phi_y+\phi_\pi k}\left[\sigma E_t\widetilde{y}_{t+1} + (1-\phi_\pi\beta)E_t\pi_{t+1} + r_t^n\right] \end{cases} \tag{41}$$

We now write the model in matrix form (note expectations are on the r.h.s.):

$$\begin{bmatrix} \widetilde{y}_t \\ \pi_t \end{bmatrix} = \Omega \begin{bmatrix} \sigma & (1-\phi_\pi\beta) \\ \sigma k & k + \beta(\sigma+\phi_y) \end{bmatrix} \begin{bmatrix} E_t\widetilde{y}_{t+1} \\ E_t\pi_{t+1} \end{bmatrix} + \begin{bmatrix} 1 \\ k \end{bmatrix} r_t^n \tag{42}$$

where $\Omega = \frac{1}{\sigma+\phi_y+\phi_\pi k}$.

In state-space form, we can write:

$$X_t = \Psi E_t X_{t+1} + \Xi V_t \tag{43}$$

where $X_t = \begin{bmatrix} \widetilde{y}_t \\ \pi_t \end{bmatrix}$, $V_t = r_t^n$, $\Psi = \Omega \begin{bmatrix} \sigma & (1-\phi_\pi\beta) \\ \sigma k & k+\beta(\sigma+\phi_y) \end{bmatrix}$, and $\Xi = \begin{bmatrix} 1 \\ k \end{bmatrix}$.

The characteristic polynomial of Ψ is given by

$$p(\lambda) = \lambda^2 + a_1\lambda + a_0 \tag{44}$$

where $a_0 = \frac{\beta\sigma}{\sigma+\phi_y+\phi_\pi k}$ and $a_1 = -\frac{k+\sigma+\beta\sigma+\beta\phi_y}{\sigma+\phi_y+\phi_\pi k}$.

We have two nonpredetermined variables $E_t\widetilde{y}_{t+1}$ and $E_t\pi_{t+1}$, so we need two explosive eigenvalues in Ψ^{-1}.

Both the eigenvalues of Ψ lie inside the unit circle if and only if the following condition holds

$$\begin{cases} |a_0| < 1 \\ |a_1| < 1 + a_0 \end{cases} \tag{45}$$

- Condition $|a_0| < 1$ implies $\phi_y + \phi_\pi k > -(1-\beta)\sigma$, which is always satisfied for $0 < \beta < 1$.
- Condition $|a_1| < 1 + a_0$ implies:

$$\phi_\pi + \frac{(1-\beta)}{k}\phi_y > 1 \tag{46}$$