Es. 2

$X \sim \text{Unif}(0,1)$

$Y \sim \text{Exp}(1)$

$$Z = \begin{cases} X & Y<1 \\ -X & Y>1 \end{cases}$$

i) $(Y=1)$ è un evento a prob. 0 poiché Y è una v.a. assolutamente continua

ii) $Z \in (-1,1)$ q.c. $\Rightarrow$ $F_Z(z) = \begin{cases} 0 & z \le -1 \\ ? & -1 < z \le 1 \\ 1 & z > 1 \end{cases}$

per $z \in (-1,1)$

$$F_Z(z) = P(Z<z, \Omega) = P(Z<z, Y<1) + P(Z<z, Y>1)$$

$$= P(X<z, Y<1) + P(-X<z, Y>1)$$

per l'indip. tra X e Y

$$= P(X<z)\,P(Y<1) + P(X>-z)\,P(Y>1).$$

Bisogna distinguere 2 casi $\begin{cases} 0<z\le 1 \\ -1<z\le 0 \end{cases}$

per $z \in (0,1)$ $\quad P(X<z) = z$

e $P(X>-z) = 1$ poiché $-z \le 0$

Quindi $F_Z(z) = z(1-e^{-1}) + 1 \cdot e^{-1}$

$= z - e^{-1}(z-1)$

per $z \in (-1,0)$ $\quad P(X<z) = 0$

e $P(X>-z) = 1-(-z) = 1+z$

Quindi $F_Z(z) = 0\,\cancel{(1-e^{-1})} + (1+z)e^{-1} = (1+z)e^{-1}$

$$\Longrightarrow \quad F_Z(z) = \begin{cases} 0 & z \le -1 \\ e^{-1}(1+z) & -1 < z \le 0 \\ z - e^{-1}(z-1) & 0 < z \le 1 \\ 1 & z > 1 \end{cases}$$

(iii) $$\mathbb{E}Z = \mathbb{E}X \, P(Y \le 1) + \mathbb{E}(-X)\, P(Y > 1)$$

$$= \frac{1}{2}(1 - e^{-1}) - \frac{1}{2} e^{-1} = \frac{1}{2} - e^{-1}$$

oppure $$f_Z(z) = \begin{cases} e^{-1} & \text{per } z(-1,0) \\ 1 - e^{-1} & z \in (0,1) \\ 0 & \text{altrove} \end{cases}$$

$$\Longrightarrow \quad \mathbb{E}Z = \int_{-1}^{0} z e^{-1} \, dz + \int_0^1 z(1 - e^{-1})\, dz$$

$$= e^{-1} \frac{z^2}{2}\Big]_{-1}^{0} + (1 - e^{-1}) \frac{z^2}{2}\Big]_0^1$$

$$= -\frac{1}{2} e^{-1} + \frac{1}{2}(1 - e^{-1}) = \frac{1}{2} - e^{-1}$$

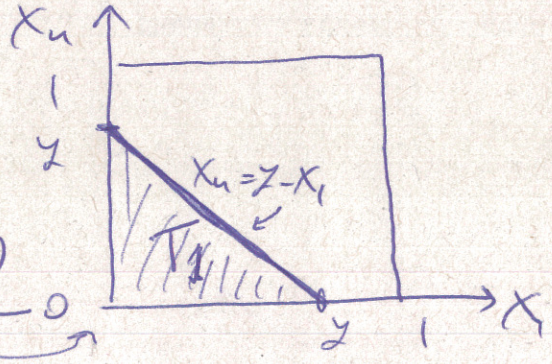
ES. 3 $$f_{X_n}(x) = \begin{cases} n x^{n-1} & x \in (0,1) \\ 0 & \text{altrove} \end{cases}$$

$X_1 \in (0,1)$, $X_n \in (0,1)$ p.c. $\Longrightarrow$ $X_1 + X_n \in (0,2)$ p.c.

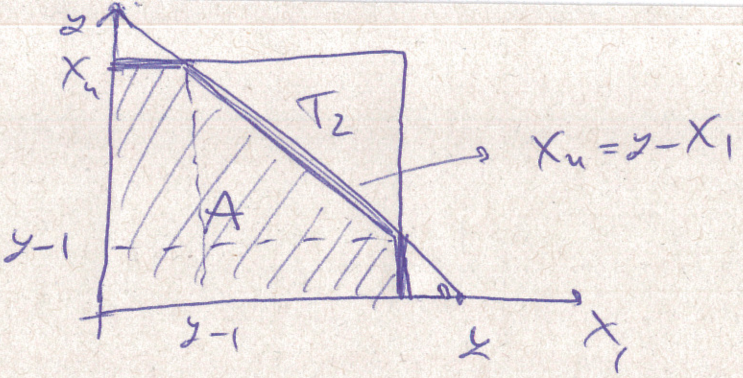
$P(X_1 + X_n < y) =$

$= P(X_n < y - X_1)$

per $y(0,1)$



pr $y \in (1,2)$



$X_1 \sim \mathrm{Unif}(0,1)$

X_n a la densité $f_{X_n}(x)$

$\Longrightarrow$ pr $y \in (0,1)$

$$p(X_1 + X_n < y) = \iint_{\{(X_1,X_n) \in T_1\}} 1\, dx_1\, n x_n^{n-1}\, dx_n$$

$$= n \int_0^y dx_1 \int_0^{y-x_1} x_n^{n-1}\, dx_n$$

$$= \cancel{n} \int_0^y dx_1 \left. \frac{x_n^n}{\cancel{n}} \right]_0^{y-x_1}$$

$$= \int_0^y (y-x_1)^n\, dx_1$$

$$= - \left. \frac{(y-x_1)^{n+1}}{n+1} \right]_0^y = \frac{y^{n+1}}{n+1}$$

pr $y \in (1,2)$

$$p(X_1 + X_n < y) = \iint_{\{(x_1,x_n) \in A\}} 1\, dx_1\, n x_n^{n-1}\, dx_n$$

$$= 1 - \iint_{\{(x_1,x_n) \in T_2\}} 1\, dx_1\, n x_n^{n-1}\, dx_n$$

$$= 1 - \int_{y-1}^1 dx_1 \int_{y-x_1}^1 n x_n^{n-1}\, dx_n$$

$$= 1 - \int_{y-1}^1 \left. x_n^n \right]_{y-x_1}^1 dx_1$$

$$= 1 - \int_{y-1}^{1} \left(1 - (y - x_1)^n\right) dx_1$$

$$= 1 - 2 + y - \frac{(y-x_1)^{n+1}}{n+1}\Bigg]_{y-1}^{1}$$

$$= -1 + y - \frac{(y-1)^{n+1}}{n+1} + \frac{1}{n+1}$$

$$F_{X_1+X_n}(y) = \begin{cases} 0 & y \le 0 \\ \dfrac{y^{n+1}}{n+1} & 0 < y \le 1 \\ \dfrac{1}{n+1} - 1 + y - \dfrac{(y-1)^{n+1}}{n+1} & 1 < y \le 2 \\ 1 & y > 2 \end{cases}$$

$$\lim_{n \to +\infty} F_{X_1+X_n}(y) = \begin{cases} 0 & y \le 0 \\ 0 & 0 < y \le 1 \\ y-1 & 1 < y \le 2 \\ 1 & y > 2 \end{cases}$$

Quindi $F_Z(y) = \begin{cases} 0 & y \le 1 \\ y-1 & 1 < y \le 2 \\ 1 & y > 2 \end{cases}$ $f_Z(y) = \begin{cases} 1 & 1 < y \le 2 \\ 0 & \text{altrove} \end{cases}$

e $X_1 + X_n \xrightarrow{d} Z$ con $Z \sim \mathrm{Unif}(1, 2)$