Ⓐ <u>Define</u>:

$$W(t) = \int_0^t ds\, \eta(s)$$

We wish to compute

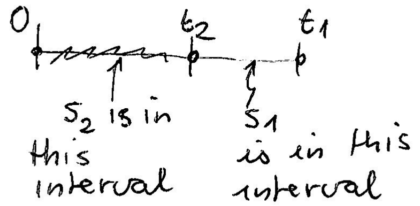
$$\begin{aligned} G(t_1, t_2) &= \langle W(t_1) W(t_2) \rangle \\ &= \int_0^{t_1} ds_1 \int_0^{t_2} ds_2 \langle \eta(s_1) \eta(s_2) \rangle \\ &= \int_0^{t_1} ds_1 \int_0^{t_2} ds_2\, \delta(s_1 - s_2) \end{aligned}$$

We should be careful about the δ- function

(a) Assume $t_1 > t_2$ $\quad \left([0, t_1] = [0, t_2] \cup [t_2, t_1] \right)$

$$= \int_0^{t_2} ds_1 \underbrace{\int_0^{t_2} ds_2\, \delta(s_1 - s_2)}_{=1} + \underbrace{\int_{t_2}^{t_1} ds_1 \int_0^{t_2} ds_2\, \delta(s_1 - s_2)}_{\text{this is zero as } s_1 \text{ can never be equal to } s_2}$$

$$= \int_0^{t_2} ds_1 = t_2$$



(b) Assume $t_2 > t_1$ $\quad \left([0,t_2] = [0,t_1] \cup [t_1,t_2] \right)$

$$= \underbrace{\int_0^{t_1} ds_1 \int_0^{t_1} ds_2 \, \delta(s_1 - s_2)}_{=1} + \underbrace{\int_0^{t_1} ds_1 \int_{t_1}^{t_2} ds_2 \, \delta(s_1 - s_2)}_{\text{this is zero. } s_1 \text{ and } s_2 \text{ go in disjoint intervals}}$$

$$= \int_0^{t_1} ds_1 = t_1$$

Therefore

$$G(t_1, t_2) = \begin{cases} t_2 & \text{if } t_2 < t_1 \\ t_1 & \text{if } t_2 > t_1 \end{cases} = \min(t_1, t_2)$$

(B) Define $(s > t)$

$$W(s,t) = \int_t^s dp \, \eta(p) = \int_0^s dp \, \eta(p) - \int_0^t dp \, \eta(p)$$

$$= W(s) - W(t)$$

$$\langle W(s_1,t) W(s_2,t) \rangle = \qquad \boxed{s_1, s_2 > t}$$

$$= \langle (W(s_1) - W(t))(W(s_2) - W(t)) \rangle$$

$$= \langle W(s_1,s_2) \rangle - \langle W(t) W(s_2) \rangle - \langle W(t) W(s_1) \rangle + \langle W(t)^2 \rangle$$

$$= \min(s_1,s_2) - \min(t, s_2) - \min(t, s_1) + t$$

$$= \min(s_1,s_2) - t - t + t = \min(s_1,s_2) - t$$

$$= \min(s_1 - t, s_2 - t)$$

[OBVIOUS: $\eta(t)$ is translation invariant]

$$W(s,t) = W(s-t)$$

Ⓒ We define

$$u(t,t+\tau) = \int_t^{t+\tau} ds\, W(s,t) = \int_t^{t+\tau} ds\, W(s-t) \qquad s-t=s'$$

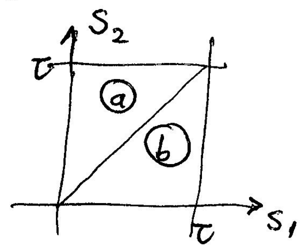
$$= \int_0^{\tau} ds'\, W(s')$$

Thus $u(t,t+\tau)$ depends only on τ

$$u(t,t+\tau) \equiv u(\tau)$$

$$\langle u(\tau)^2\rangle = \int_0^{\tau} ds_1 \int_0^{\tau} ds_2 \, \langle W(s_1) W(s_2)\rangle =$$

$$= \int_0^{\tau} ds_1 \int_0^{\tau} ds_2 \min(s_1, s_2)$$



In ⓐ $s_2 > s_1$

In ⓑ $s_1 > s_2$

$$\langle u(\tau)^2\rangle = \int_{ⓐ} ds_1 ds_2\, s_1 + \int_{ⓑ} ds_1 ds_2\, s_2$$

$$= \int_0^{\tau} ds_1 \int_{s_1}^{\tau} ds_2\, s_1 + \int_0^{\tau} ds_1 \int_0^{s_1} ds_2\, s_2 = \frac{T^3}{6} + \frac{T^3}{6} = \frac{T^3}{3}$$

Ⓓ

$$\langle u(\tau) W(\tau)\rangle = \int_0^{\tau} ds\, \langle W(s) W(\tau)\rangle =$$

$$= \int_0^{\tau} ds \min(s,\tau) = \int_0^{\tau} ds\, s = \frac{\tau^2}{2}$$

Recap:

We have introduced

Replace dtau with tau

$$W(\tau) = \int_t^{t+d\tau} \eta(s)\,ds$$

$$u(\tau) = \int_t^{t+d\tau} W(s)\,ds$$

TRANSLATION INVARIANCE implies no t-dependence

They are both Gaussian variables as they are "sum" of Gaussian noise

They have zero mean $\langle W(\tau)\rangle = \langle u(\tau)\rangle = 0$ since $\langle \eta(s)\rangle = 0$

Their variances are

$$\langle W(\tau)^2\rangle = \tau$$

$$\langle u(\tau)^2\rangle = \frac{1}{3}\tau^3$$

They are correlated since

$$\langle u(\tau)W(\tau)\rangle = \frac{\tau^2}{2}$$

We now define two new Gaussian variables $\xi(t)$ and $\theta(t)$ such that

$$\langle \xi(t) \rangle = \langle \theta(t) \rangle = 0$$ zero = mean

$$\langle \xi(t)^2 \rangle = \langle \theta(t)^2 \rangle = 1$$ one = variance

$$\langle \xi(t)\theta(t) \rangle = 0$$ <u>un</u> correlated

They are defined as

$$\begin{cases} u(\tau) = \alpha\xi(\tau) + \beta\theta(\tau) \\ W(\tau) = \gamma\xi(\tau) \end{cases}$$

$$\langle W(\tau)^2 \rangle = \gamma^2 \langle \xi(\tau)^2 \rangle = \gamma^2$$

$$\langle u(\tau)^2 \rangle = \langle [\alpha\xi(\tau) + \beta\theta(\tau)]^2 \rangle =$$
$$= \alpha^2 \langle \xi^2 \rangle + \beta^2 \langle \theta^2 \rangle + 2\alpha\beta \langle \xi\theta \rangle = \alpha^2 + \beta^2$$

$$\langle uW \rangle = \langle (\alpha\xi + \beta\theta)\gamma\xi \rangle =$$
$$= \alpha\gamma \langle \xi^2 \rangle + \beta\gamma \langle \theta\xi \rangle = \alpha\gamma$$

Therefore

$$\begin{cases} \gamma^2 = \tau \\ \alpha^2 + \beta^2 = \tau^3/3 \\ \alpha\gamma = \tau^2/2 \end{cases}$$

$$\gamma = \sqrt{\tau}$$
$$\alpha = \frac{\tau^2}{2\gamma} = \frac{\tau^{3/2}}{2}$$
$$\beta^2 = \frac{\tau^3}{3} - \alpha^2 = \frac{\tau^3}{3} - \frac{\tau^3}{4} = \frac{\tau^3}{12} \qquad \beta = \frac{\tau^{3/2}}{2\sqrt{3}}$$

$$\begin{cases} W(\tau) = \sqrt{\tau}\,\xi(\tau) \\ u(\tau) = \frac{1}{2}\tau^{3/2}\left[\xi(\tau) + \frac{1}{\sqrt{3}}\theta(\tau)\right] \end{cases}$$