# STOCHASTIC EQUATIONS

We consider the equation

$$\frac{dr}{dt} = \lambda F(r) + \alpha \eta(t) \qquad r(t=0) = r_0$$

where $\eta(t)$ is a random variable [noise]

We assume Gaussian uncorrelated noise

$$\langle \eta(t) \rangle = 0 \qquad \langle \eta(t)\eta(s) \rangle = \delta(t-s)$$

NOTE: DIRAC DELTA FUNC

We wish to define a numerical algorithm for this equation.

The verlet idea

$$r(t+\Delta t) = r(t) + v(t)\Delta t$$

$$r(t+\Delta t) = r(t) + \Delta t\left(\lambda F(t) + \alpha(\eta(t))\right)$$

UNFORTUNATELY: THIS SCHEME IS <u>WRONG</u>

IN The Verlet approach one assumes that all quantities in the equation are SMOOTH in $t$.

THE NOISE IS <u>NOT</u> SMOOTH

## THE WAY OUT

We integrate the equation between $t$ and $t+\Delta t$

$$\int_t^{t+\Delta t} \frac{dr}{dt}\, dt = \lambda \int_t^{t+\Delta t} F(r(t))\, dt + \alpha \int_t^{t+\Delta t} \eta(t)\, dt$$

$$r(t+\Delta t) - r(t) = \lambda \int_t^{t+\Delta t} F(r(t))\, dt + \alpha \int_t^{t+\Delta t} \eta(t) dt$$

Now, the force $F(r)$ depends smoothly on $r$:

$$\int_t^{t+\Delta t} F(r(s))\, ds \approx F(r(t))\, \Delta t$$

For the noise term we define $\xi(t) = \int_t^{t+\Delta t} \eta(s)\, ds$

$\xi(t)$ is a "sum" (integral) of Gaussian variables of zero mean.

Therefore $\xi(t)$ is a Gaussian variable of zero mean

$$\langle \xi(t) \rangle = 0$$

VARIANCE:

$$\langle \xi(t)^2 \rangle = \langle \int_t^{t+\Delta t} \eta(s_1)\, ds_1 \int_t^{t+\Delta t} \eta(s_2)\, ds_2 \rangle$$

$$= \int_t^{t+\Delta t} ds_1 \int_t^{t+\Delta t} ds_2 \langle \eta(s_1)\eta(s_2) \rangle$$

$$= \int_t^{t+\Delta t} ds_1 \int_t^{t+\Delta t} ds_2 \, \delta(s_1 - s_2) = \int_t^{t+\Delta t} ds_1 = \Delta t$$

Thus: $\langle \xi(t) \rangle = 0 \qquad \langle \xi^2(t) \rangle = \Delta t$

We thus define $\rho(t) = \frac{1}{\sqrt{\Delta t}} \xi(t)$

$$r(t+\Delta t) = r(t) + \lambda F(t) \Delta t + \alpha \sqrt{\Delta t} \, \rho(t)$$

$\rho(t)$: Gaussian variable with variance = 1

Note the $\sqrt{\Delta t}$ DEPENDENCE

The equation can be seen as a Markov process, in which the probability of arriving in a given point $r(t+\Delta t)$ depends on the probability distribution of $\rho(t)$

IMPLEMENTATION

Given $r(t)$

a) compute $F(r(t)) \equiv F(t)$

b) extract Gaussian variable with variance 1 $\rho$

c) $r(t+\Delta t) = r(t) + \lambda F(t) \Delta t + \alpha \sqrt{\Delta t} \, \rho(t)$

ITERATE