MOMENT-GENERATING FUNCTION FOR A GAUSSIAN DISTRIBUTION

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$$

mean : 0
variance : σ^2

$$\langle e^{itx} \rangle = \frac{1}{\sqrt{2\pi}\,\sigma} \int dx\, e^{-x^2/2\sigma^2} e^{itx}$$

We now change variable ; $x \longrightarrow y = \frac{1}{\sqrt{2}\,\sigma}(x-b)$

b is a parameter that is fixed below

$$-\frac{x^2}{2\sigma^2} + itx = -\frac{1}{2\sigma^2}(\sqrt{2}\sigma y + b)^2 + it(\sqrt{2}\sigma y + b)$$

$$= -y^2 - \sqrt{2}y\left(\frac{b}{\sigma} - it\sigma\right) - \frac{b^2}{2\sigma^2} + itb$$

Now we fix b, requiring

$$\frac{b}{\sigma} - it\sigma = 0 \implies b = it\sigma^2$$

$$-\frac{x^2}{2\sigma^2} + itx = -y^2 - \frac{1}{2\sigma^2}(it\sigma^2)^2 + it(it\sigma^2) = -y^2 - \frac{t^2\sigma^2}{2}$$

Thus

$$\langle e^{itx} \rangle = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} \sqrt{2}\sigma\, dy\, e^{-y^2} e^{-t^2\sigma^2/2}$$

[there is also a hidden shift in the complex plane]

$$= \frac{1}{\sqrt{\pi}} e^{-t^2\sigma^2/2} \int_{-\infty}^{+\infty} dy\, e^{-y^2} = e^{-t^2\sigma^2/2}$$

Gaussian distribution; $P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$
zero mean $\langle x \rangle = 0$
variance $\langle x^2 \rangle - \langle x \rangle^2 = \sigma^2$

We want to compute the distribution of

$$X = \sum_{i=1}^{N} x_i = x_1 + \dots x_N \qquad \left[\begin{array}{l} x_i \text{ Gaussian} \\ \text{with variance } \sigma_i^2 \end{array} \right]$$

We compute the moment generating function

$$\langle e^{itX} \rangle \qquad t \text{ is a free parameter}$$

$$= \int dx_1 .. dx_N \, e^{itX} P(x_1) \dots P(x_n)$$

$$= \int dx_1 e^{itx_1} P(x_1) \int dx\, e^{itx_2} P(x_2) \dots \int dx\, e^{itx_n} P(x_n)$$

$$= e^{-t^2\sigma_1^2/2} e^{-t^2\sigma_2^2/2} \dots e^{-t^2\sigma_N^2/2}$$

$$= e^{-t^2\Sigma^2/2} \qquad \Sigma^2 = \sum_{i=1}^{N} \sigma_i^2 = \sigma_1^2 + \dots + \sigma_N^2$$

Thus, X is a Gaussian variable of variance Σ^2.

THE BASIC INGREDIENT : THE NOISE

In the simplest stochastic equations we consider a Gaussian uncorrelated noise

Formally, its probability distribution is

$$P(\eta) \approx \exp\left(-\frac{1}{2}\int_a^b dt\, \eta(t)^2\right)$$

[a, b may be taken $-\infty, +\infty$, resp.]

To understand the meaning of this distribution, let us discretize the integral [step $\Delta t = \epsilon$]

$$P(\eta) \approx \exp\left(-\frac{\epsilon}{2}\sum_i \eta(t_i)^2\right)$$

This is the sum of uncorrelated Gaussian variables of variance $1/\epsilon$ (i.e diverging for $\epsilon \to 0$)

We wish to compute $G(t-s) = \langle \eta(t)\eta(s)\rangle$

We consider a smooth function $f(x)$

$$\int dt\, f(t)\, G(t-s) = \int dt\, f(t)\, \langle \eta(t)\eta(s)\rangle = \left(\text{discretization}\right)$$

$$= \sum_i \epsilon f(t_i) \underbrace{\langle \eta(t_i)\eta(s)\rangle}_{\substack{=0 \text{ if } s \neq t_i \\ = 1/\epsilon \text{ if } s = t_i}} =$$

$$= \sum_i \epsilon f(t_i)\, \delta(s-t_i)\, 1/\epsilon = f(s)$$

It implies $\quad G(t-s) = \delta(t-s)$