• SUPPLEMENTARY MATERIAL

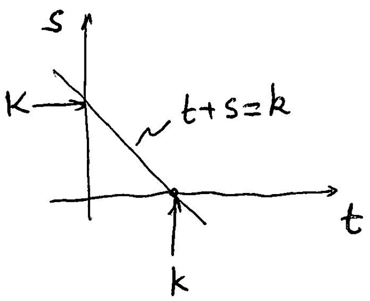
(a) Relation "Beta" (Euler's B-Integral)

$$\int_0^1 dy\, y^n (1-y)^m = \frac{n!\, m!}{(n+m+1)!}$$

PROOF:

Note that $\int_0^\infty dt\, e^{-t} t^n = n!$ (Γ-integral; it can be easily proved by integration by parts)

$$\underbrace{\int_0^\infty dt\, e^{-t} t^n \int_0^\infty ds\, e^{-s} s^m}\; = n!\, m!$$



We introduce a new variable

$k = t+s \qquad s = k-t$

$$\int_0^\infty dt \int_0^\infty ds \cdots = \int_0^\infty dk \int_0^k dt \cdots$$

$$n!\, m! = \int_0^\infty dk \int_0^k dt\, e^{-k}\, t^n (k-t)^m \qquad \text{Now} \quad t = kx$$

$$= \underbrace{\int_0^\infty dk\, k^{n+m+1} e^{-k}}_{(n+m+1)!} \int_0^1 dx\, x^n (1-x)^m$$

It gives $\int_0^1 dx\, x^n (1-x)^m = \dfrac{n!\, m!}{(m+n+1)!}$

(6) Relation for exponentials of operators

$$e^{A} B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]]$$

$$+ \ldots \frac{1}{n!} \underbrace{[A,[A, \ldots [A,B]]] \ldots]}_{n \text{ nested commutators}} + \ldots$$

Proof

$$f(t) = e^{tA} B e^{-tA}$$

$$\frac{df}{dt} = e^{tA} A B e^{-tA} + e^{tA} B(-A) e^{-tA} = e^{tA} [A,B] e^{-tA}$$

$$\frac{d^2 f}{dt^2} = \frac{d}{dt} e^{tA} [A,B] e^{-tA} = e^{tA} [A,[A,B]] e^{-tA}$$

$$\frac{d^3 f}{dt^3} = \frac{d}{dt} e^{tA} [A,[A,B]] e^{-tA} = e^{tA} [A,[A,[A,B]]] e^{-tA}$$

$$\vdots$$

Now

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n f}{dt^n}\right|_{t=0} t^n$$

$$= B + [A,B]\, t + \frac{1}{2}[A,[A,B]]\, t^2 + \ldots$$

The result is obtained by setting $t=1$

c) The derivative of an exponential

$$\frac{d}{dt} e^{At} = A e^{At} \quad \text{correct}$$

BUT $\frac{d}{dt} e^{A(t)} \neq A'(t) e^{A(t)}$ unless $\begin{cases} [A, A'] = 0 \\ A(t) = a_0 t \text{ is ok} \end{cases}$

GENERAL FORMULA : $\frac{d}{dt} e^{A(t)} = \int_0^1 dx \, e^{(1-x)A(t)} \frac{dA}{dt} e^{xA(t)}$

PROOF :

$$e^{A(t)} = \sum_{n=0}^{\infty} \frac{1}{n!} A(t)^n$$

$$= 1 + A + \frac{1}{2} A^2 + \frac{1}{3!} A^3 + \dots$$

$$\frac{d}{dt} e^{A(t)} = A' + \frac{1}{2}(A'A + AA') + \frac{1}{3!}(A'A^2 + AA'A + AAA')$$

$$+ \dots \frac{1}{n!} \sum_{k=0}^{n-1} A^k A' A^{n-k-1} + \dots$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n!} A^k A' A^{n-k-1} \qquad n = p+1$$

$$= \sum_{p=0}^{\infty} \sum_{k=0}^{p} \frac{1}{\underbrace{(p+1)!}_{[k+(p-k)+1]}} A^k A' A^{p-k}$$

In the sum we have all terms $A^n A' A^m$ with coefficient $\frac{1}{(n+m+1)!}$

$$\frac{d}{dt} e^{A(t)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(n+m+1)!} A^n A' A^m$$

Now we consider the integral

$$\int_0^1 dx\, e^{(1-x)A(t)} \frac{dA}{dt} e^{xA(t)} = \quad \left(\text{We expand the exponentials}\right)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_0^1 dx \frac{(1-x)^n}{n!} A^n A' \frac{x^m}{m!} A^m$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!} \frac{1}{m!} A^n A' A^m \underbrace{\int_0^1 dx\, (1-x)^n x^m}_{\frac{n!\,m!}{(n+m+1)!}}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(n+m+1)!} A^n A' A^m \quad \text{which is equal to } \frac{d}{dt} e^A$$

The general formula implies

$$\frac{d}{dt} e^{A(t)} = e^{A(t)} \int_0^1 dx\, e^{-xA(t)} \frac{dA}{dt} e^{xA(t)}$$

$$= e^{A(t)} \int_0^1 dx \left[A' - x[A,A'] + \frac{x^2}{2}[A,[A,A']] - \frac{x^3}{6}[A,[A,[A,A']]] + \dots \right)$$

$$= e^{A(t)} \left[A' - \frac{1}{2}[A,A'] + \frac{1}{3!}[A,[A,A']] - \frac{1}{4!}[A,[A,[A,A']]] + \dots \right]$$

Formula of Baker-Campbell-Hausdorff

$$e^{tA}e^{tB} = e^{c(t)}$$

We wish to compute $C(t)$ as a power series in t

METHOD

Note $e^{-C(t)} = e^{-tB}e^{-tA}$

indeed
$$\begin{cases} e^{-tB}e^{-tA}e^{ct} = e^{-tB}e^{-tA}e^{tA}e^{tB} = 1 \\ e^{-c(t)}e^{c(t)} = 1 \end{cases}$$

$$\Longrightarrow e^{-c(t)} = e^{-tB}e^{-tA}$$

Second observation

$$e^{-tB}e^{-tA}\frac{d}{dt}(e^{tA}e^{tB}) =$$

Remember $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$

$$= e^{-tB}e^{-tA}\left[Ae^{At}e^{Bt} + e^{At}e^{Bt}B\right]$$

$$= e^{-tB}\underbrace{e^{-tA}Ae^{At}}_{A}e^{Bt} + \underbrace{e^{-tB}e^{-tA}e^{At}e^{Bt}}_{\mathbb{1}}B$$

$$=$$

$$= B + e^{-tB}Ae^{tB} = B + A - [B,A]t + \frac{1}{2}[B,[B,A]]t^2 + \dots$$

it is equal to

$$e^{-c(t)}\frac{d}{dt}e^{c(t)} = c' - \frac{1}{2}[c,c'] + \frac{1}{3!}[c,[c,c']] + \dots$$

We can expand

$$c(t) = c_1 t + c_2 t^2 + c_3 t^3 + \dots$$

$$c'(t) = c_1 + 2c_2 t + 3c_3 t^2 + \dots$$

$$[c, c'] = [c_1 t + c_2 t^2, c_1 + 2c_2 t] + O(t^3)$$

$-[c_1, c_2]$

$$= \cancel{[c_1, c_1] t} + 2[c_1, c_2] t^2 + [c_2, c_1] t^2$$

$$= [c_1, c_2] t^2 + O(t^3)$$

Comparing

$$e^{-c(t)} \frac{d}{dt} e^{c(t)} = c' - \frac{1}{2}[c, c'] + O(t^3)$$

$$= c_1 + 2c_2 t + 3c_3 t^2 - \frac{1}{2}[c_1, c_2] t^2 + O(t^3)$$

$c_1 = A + B$ (terms of order t^0)

$+2c_2 = -[B, A]$ $\quad$ $c_2 = \frac{1}{2}[A, B]$ (terms of order t^1)

$3c_3 - \frac{1}{2}[c_1, c_2] = \frac{1}{2}[B, [B, A]]$ (terms of order t^2)

$$c_3 = \frac{1}{6}[c_1, c_2] + \frac{1}{6}[B, [B, A]]$$

$$= \frac{1}{6}[A + B, \frac{1}{2}[A, B]] - \frac{1}{6}[B, [A, B]]$$

$$= \frac{1}{12}[A, [A, B]] + \frac{1}{12}[B, [A, B]] - \frac{1}{6}[B, [A, B]]$$

$$= \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B]]$$

It is clear from the derivation that $c_4, c_5, \dots$ are nested commutators of A and B.

A useful formula

$$e^{\epsilon\alpha B}\, e^{\epsilon A}\, e^{\epsilon(1-\alpha)B}$$

$\alpha \equiv$ number
$\epsilon =$ small parameter

We use $\quad e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}$

$$e^{\epsilon\alpha B}\, e^{\epsilon A} = \exp\left(\epsilon A + \epsilon\alpha B + \frac{1}{2}[\epsilon\alpha B, \epsilon A] + \dots\right)$$
$$= \exp\left(\epsilon A + \epsilon\alpha B - \frac{\epsilon^2}{2}\alpha[A,B] + \dots\right)$$

$$\left(e^{\epsilon\alpha B}\, e^{\epsilon A}\right)\exp[\epsilon(1-\alpha)B]$$

$$= \exp\Big[\epsilon A + \epsilon\alpha B - \frac{\epsilon^2}{2}\alpha[A,B] + \dots + \epsilon(1-\alpha)B + \frac{1}{2}[\epsilon A + \epsilon\alpha B, \epsilon(1-\alpha)B] + \dots\Big]$$

$$= \exp\Big[\epsilon A + \epsilon\alpha B + \epsilon(1-\alpha)B - \frac{\epsilon^2}{2}\alpha[A,B] + \frac{\epsilon^2}{2}(1-\alpha)[A,B] + \dots\Big]$$

$$= \exp\Big[\epsilon(A+B) + \frac{\epsilon^2}{2}(1-2\alpha)[A,B] + \underbrace{\dots}_{O(\epsilon^3)}\Big]$$

If $\alpha = 1/2$

$$e^{\epsilon B/2}\, e^{\epsilon A}\, e^{\epsilon B/2} = e^{\epsilon(A+B) + O(\epsilon^3)}$$