CONSERVATION OF PHASE SPACE VOLUME

Simple case: one particle in one dimension

$$\begin{cases} r(t+\Delta t) = r(t) + \frac{1}{m} p(t)\Delta t + \frac{1}{2m} F(t)\Delta t^2 \qquad F(t) \equiv F(r(t)) \\ p(t+\Delta t) = p(t) + \frac{1}{2}\left[F(t) + F(t+\Delta t)\right]\Delta t \end{cases}$$

We want to prove that the jacobian

$$J = \begin{pmatrix} \frac{\partial r(t+\Delta t)}{\partial r(t)} & \frac{\partial r(t+\Delta t)}{\partial p(t)} \\ \frac{\partial p(t+\Delta t)}{\partial r(t)} & \frac{\partial p(t+\Delta t)}{\partial p(t)} \end{pmatrix}$$

has determinant 1 : $\det J = 1$

BEWARE: $F(t+\Delta t)$ depends on $r(t+\Delta t)$, which in turn depends on $r(t)$ and $p(t)$

$$\frac{\partial r(t+\Delta t)}{\partial r(t)} = 1 + \frac{\Delta t^2}{2m} F'(t) \qquad F'(t) = \left.\frac{\partial F(r)}{\partial r}\right|_{r=r(t)}$$

$$\frac{\partial r(t+\Delta t)}{\partial p(t)} = \frac{\Delta t}{m} \qquad F'(t+\Delta t) = \left.\frac{\partial F}{\partial r}\right|_{r=r(t+\Delta t)}$$

$$\frac{\partial p(t+\Delta t)}{\partial p(t)} = 1 + \frac{\Delta t}{2} F'(t+\Delta t) \frac{\partial r(t+\Delta t)}{\partial p(t)}$$

$$\frac{\partial p(t+\Delta t)}{\partial r(t)} = \frac{\Delta t}{2} F'(t) + \frac{\Delta t}{2} F'(t+\Delta t) \frac{\partial r(t+\Delta t)}{\partial r(t)}$$

$$J = \frac{\partial r(t+\Delta t)}{\partial r(t)} \frac{\partial p(t+\Delta t)}{\partial p(t)} - \frac{\partial r(t+\Delta t)}{\partial p(t)} \frac{\partial p(t+\Delta t)}{\partial r(t)} =$$

$$= \frac{\partial r(t+\Delta t)}{\partial r(t)} \left[1 + \frac{\Delta t}{2} F'(t+\Delta t) \frac{\Delta t}{m} \right] -$$

these two terms cancel

$$- \frac{\Delta t}{m} \left[\frac{\Delta t}{2} F'(t) + \frac{\Delta t}{2} F'(t+\Delta t) \frac{\partial r(t+\Delta t)}{\partial r(t)} \right]$$

$$= \frac{\partial r(t+\Delta t)}{\partial r(t)} - \frac{\Delta t^2}{2m} F'(t) =$$

$$= 1 + \frac{\Delta t^2}{2m} F'(t) - \frac{\Delta t^2}{2m} F'(t) = 1$$

The jacobian is 1 for every Δt