Some math on $C_f(n,m)$

We wish to show how $C_f(n,m)$ can be related to the transition matrix P. We will use the same notations we used in the previous addendum, where we presented a mathematical proof that the bias is of order $1/N$ and obtained an exact expression in terms of the transition matrix.

In terms of the transition matrix we can write (for $n < m$)

$$C_f(n,m,X_0) = \langle (f(X_n) - F)(f(X_m) - F) \rangle_{MC,0} = \sum_{xy} (f(x) - F)(f(y) - F) P^n_{0x} P^{m-n}_{xy} .$$

where we set $F = \langle f(x) \rangle_\pi$.

Now we assume that we start in equilibrium. Rigorously, this means that we pick the starting point X_0 with probability π and average over the initial conditions. In practice, it corresponds to consider a typical configuration X_0. Indeed, if we choose X_0 with probability π, we will only obtain typical configuration.
Therefore, the equilibrium autocorrelation function is

$$C_{f,\mathrm{eq}}(n,m) = \sum_x \pi_x C_f(n,m,x)$$

We obtain

$$C_{f,\mathrm{eq}}(n,m) = \sum_z \pi_z \sum_{xy} (f(x)-F)(f(y)-F) P^n_{zx} P^{m-n}_{xy} = \sum_{xy} \pi_x (f(x)-F)(f(y)-F) P^{m-n}_{xy},$$

In the last step we have used repeatedly the stationarity condition:

$$\sum_z \pi_z P^n_{zx} = \sum_{zy} \pi_z P_{zy} P^{n-1}_{yx} = \sum_y (\sum_z \pi_z P_{zy}) P^{n-1}_{yx} = \sum_y \pi_y P^{n-1}_{yx} = \ldots = \pi_x$$

The dependence on the starting time disappears, so that the autocorrelation function depends only on the difference $m-n$.

We obtain

$$\begin{aligned}
&\sum_{xy} \pi_x f(x)(f(y)-F)P_{xy}^{m-n} - F\sum_{xy} \pi_x P_{xy}^{m-n}(f(y)-F) \\
&\quad = \sum_{xy} \pi_x f(x)(f(y)-F)P_{xy}^{m-n} - F\sum_{y} \pi_y (f(y)-F) \\
&\quad = \sum_{xy} \pi_x f(x)(f(y)-F)P_{xy}^{m-n} \\
&\quad = \sum_{xy} \pi_x f(x) f(y) P_{xy}^{m-n} - F\sum_{x} \pi_x f(x) \sum_{y} P_{xy}^{m-n} = \sum_{xy} \pi_x f(x) f(y) P_{xy}^{m-n} - F^2
\end{aligned}$$

Let us comment on the different steps.
a) The second line is obtained from the first using the relation derived above $\sum_x \pi_x P_{xy}^{n-m} = \pi_y$.
b) The last term in the second line vanishes, since $\sum_y \pi_y f(y) = F$ and $\sum_y \pi_y F = F$.
c) In the last line we use $\sum_y P_{xy}^{m-n} = 1$.

Defining the projector $\Pi_{xy} = \pi_y$ we have

$$
\begin{aligned}
\sum_{xy} \pi_x f(x) f(y) P_{xy}^{m-n} - F^2 &= \sum_{xy} \pi_x f(x) f(y) P_{xy}^{m-n} - \sum_{xy} \pi_x \pi_y f(x) f(y) = \\
&= \sum_{xy} \pi_x f(x) f(y) [P_{xy}^{m-n} - \Pi_{xy}] \\
&= \sum_{xy} \pi_x f(x) f(y) [P - \Pi]_{xy}^{m-n}
\end{aligned}
$$

The derivation of the relation $P^{m-n} - \Pi = (P - \Pi)^{m-n}$ is given in the notes on the bias.

As we already discussed all eigenvalues of $P - \Pi$ lie in the unit circle (this is correct for a finite system; for an infinite one some caveats are needed). Hence, the autocorrelation function decays exponentially as $|\lambda_2|^{m-n}$ as $m - n \to \infty$, where λ_2 is the second largest (in absolute value) eigenvalue of P.