ESTIMATING THE VARIANCE

To compute the error σ, $\sigma = \frac{1}{\sqrt{N}} \left(Var_\pi f \right)^{1/2}$,
we must estimate $Var_\pi f$

We estimate it from the data as

$$EST\,(Var_\pi f) = \overline{f^2} - (\bar{f})^2 =$$
$$= \frac{1}{N} \sum_{i=1}^{N} f(x_i)^2 - \left[\frac{1}{N} \sum_{i=1}^{N} f(x_i) \right]^2$$

Now

$$\langle \overline{f^2} \rangle_{MC} = \frac{1}{N} \sum_{i=1}^{N} \langle f(x_i)^2 \rangle_{MC} = \langle f^2 \rangle_\pi$$

Therefore

$$\langle \overline{f^2} - (\bar{f})^2 \rangle_{MC} = \langle f^2 \rangle_\pi - \left[\frac{1}{N} \langle f^2 \rangle_\pi + \frac{N-1}{N} \langle f \rangle_\pi^2 \right]$$
$$= \frac{N-1}{N} \langle f^2 \rangle_\pi - \frac{N-1}{N} \langle f \rangle_\pi^2$$
$$= \left(\frac{N-1}{N} \right) Var_\pi f$$

- $\langle \overline{f^2} - (\bar{f})^2 \rangle_{MC}$ is a <u>BIASED</u> Estimator

$$\langle \overline{f^2} - (\bar{f})^2 \rangle_{MC} - Var_\pi f = \underbrace{- \frac{1}{N} Var_\pi f}_{\text{BIAS}}$$

The bias scales like $\frac{1}{N}$ [GENERAL RESULT]

An unbiased estimator is

$$\frac{N}{N-1}\left(\overline{f^2}-(\bar{f})^2\right)$$

Thus, to estimate the error we use

$$\sigma^2 = \frac{1}{N}\cdot\frac{N}{N-1}\left(\overline{f^2}-(\bar{f})^2\right) =$$

$$= \frac{1}{N-1}\left(\overline{f^2}-(\bar{f})^2\right)$$

The only difference is $N \longrightarrow N-1$.

It is important to take the bias into account only if N is very small

Note:

errors $\sim \frac{1}{\sqrt{N}}$

bias $\sim \frac{1}{N}$

bias subleading with respect to the error.