THE MONTE Carlo method is a method that is used to generate elements from a given probability distribution.

We use it to compute integrals

EXAMPLE. Suppose we wish to compute

$$I = \int_0^L dx\, f(x)$$

The MC strategy: rewrite I as an average over a probability distribution.

Define: $\pi(x) = \frac{1}{L}$ in $[0, L]$

$\pi(x)$ is a probability density in $[0, L]$

(i) $\pi(x) \geq 0$

(ii) $\int_0^L dx\, \pi(x) = 1$

Then, we rewrite

$$I = \int_0^L dx\, f(x) = L \int_0^L \frac{dx}{L} f(x) = L \int_0^L \pi(x) f(x) =$$

$$= L \langle f \rangle_\pi$$

The MC method is used to compute $\langle f \rangle_\pi$

Algorithm to compute $\langle f \rangle_\pi$

Input: N = number of iterations

(pseudo code)

```
Sum = 0.
REPEAT  N  TIMES
        Generate X with probability π(x)
        Sum = Sum + f(X)
END REPEAT
<f>_π = (1/N) Sum
```

The reason why this algorithm works is the sample-mean theorem

$$\frac{1}{N}\sum_{i=1}^{N} f(X_i) \longrightarrow \langle f \rangle_\pi \quad \text{for } N \to \infty$$

X_i extracted with probability (density) $\pi(x)$

THIS RESULT IS TRUE for any PROBABILITY $\pi(x)$

The practical problem: generate numbers X_i with probability $\pi(x)$

$\boxed{\text{Monte Carlo algorithms are meant to solve this problem.}}$

THE BASIC ROUTINE

All MC programs include a routine (we call it RAN( )) that generates <u>uniform</u> random numbers in [0,1]

RAN( ) generates numbers X_i such that

$$\text{prob}(a < X_i < a+\delta) = \delta \quad \text{with} \quad 0 \le a < a+\delta \le 1$$

The second (VERY IMPORTANT) property of RAN( ):

There are NO CORRELATIONS between two different random numbers X_i, X_j $(i \neq j)$

$$\text{prob}(a < X_i < a+\delta \text{ AND } b < X_j < b+\delta) = \delta^2$$

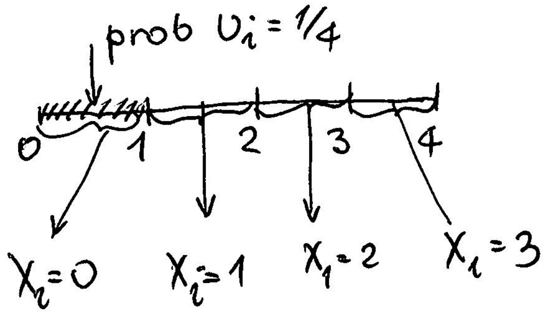
- This routine can be used to generate numbers X_i uniformly distributed in $[0,L]$

$$X_i = L * \text{RAN}(\ )$$

- This routine can be used to generate an INTEGER number uniformly in $0,\dots,L$.

$$X_i = \text{floor}[\overbrace{(L+1) * \text{RAN}(\ )}^{U_i}]$$

$\boxed{L=3}$ $\boxed{0 \le U_i < 4}$



all with probability $\frac{1}{4}$.

AN IMPORTANT COMMENT

Most random number generators (RNG) generate uniform random numbers in $[0,1[$, NOT $[0,1]$

Note the difference

$RAN(\,)=1$ never occurs

This is important in many contexts

If $RAN(\,)=1$ is generated the algorithm to generate an integer DOES NOT work

$$RAN(\,)=1 \rightarrow U_1=(L+1) \longrightarrow X_i=\text{floor}\,(L+1)=L+1$$

X_i DOES NOT belong to $0 \dots L$

(The program may crash)