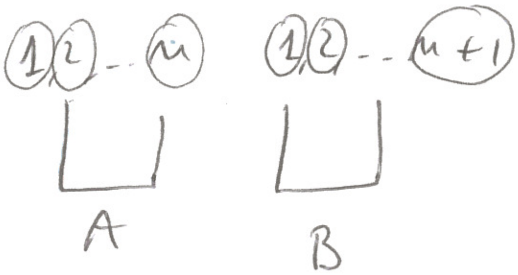
SOLUZIONI   APP. STRAORDINARIO   10-11-'16

1)



i) N_A = "n° estratto da A" N_B = "n° estratto da B"

$$P(N_A<N_B) = P\left(N_A<N_B, \bigcup_{r=1}^{n} N_A=r\right)$$

$$= \sum_{r=1}^{n} P(r<N_B, N_A=r)$$

per indipendenza $$= \sum_{r=1}^{n} P(N_B>r)\,P(N_A=r)$$

$$= \sum_{r=1}^{n} \frac{1}{n}\,\frac{n+1-r}{n+1} \qquad n+1-r=l$$

$$= \frac{1}{n(n+1)} \sum_{l=1}^{n} l = \frac{1}{\cancel{n(n+1)}}\,\frac{\cancel{n(n+1)}}{2} = \frac{1}{2}$$

ii) E = "estraggo ⑦ da B"

X_A = "n° palline estratte da A"

$$P(E) = P\left(E, \bigcup_{k=1}^{n} (X_A=k)\right)$$

$$= \sum_{k=1}^{n} P(E\mid X_A=k)\,P(X_A=k)$$

"prob. di estrarre ⑦ da un'urna in cui ho $n+1+k$ palline di cui 2 ⑦"

$$= \sum_{k=1}^{n} \frac{2}{n+k+1} \cdot P(\overbrace{\bar{I}, \bar{I}, \ldots \bar{I}}^{k-1}, S)$$

dove $\bar{I}$ = "insuccesso" = "esce n° $\neq$ 7"

S = "successo" = "esce 7"

$$= \sum_{k=1}^{n} \frac{2}{n+k+1} \cancel{\frac{n-1}{n}} \cdot \cancel{\frac{n-2}{n-1}} \cdot \cancel{\frac{n-3}{n-2}} \cdots \cancel{\frac{n-(k-1)}{n-(k-2)}} \cdot \cancel{\frac{1}{n-k+1}}$$

$$= \frac{2}{n} \sum_{k=1}^{n} \frac{1}{n+k+1}$$

2) i) $-\int_0^1 \lg(x^k)\, dx$

$\lg(x^k) = y$

$e^y = x^k$

$x = e^{y/k}$

$dx = \frac{1}{k} e^{y/k}\, dy$

$$= -\int_{-\infty}^{0} y \frac{e^{y/k}}{k}\, dy$$

$-y = w \quad y = -w$

$dy = -dw$

$$= \int_0^{+\infty} \frac{w e^{-w/k}}{k}\, dw$$

$= \mathbb{E}W$ dove $W \sim Exp(\frac{1}{k})$

$= k = 1$

ii) $Z_n = -\lg\left[\left(\prod_{i=1}^{n} X_i\right)^{1/n}\right]$

$$= -\frac{1}{n} \lg\left(\prod_{i=1}^{n} X_i\right) = -\frac{1}{n} \sum_{i=1}^{n} \lg(X_i)$$

Sia $Z_n = \frac{1}{n}\sum_{i=1}^{n}\left(-\lg(X_i)\right) = \frac{1}{n}\sum_{i=1}^{n} Y_i$ (3)

poiché le X_i sono i.i.d. e $\mathbb{E}X_i < \infty$ lo sono anche le Y_i

$$\Rightarrow \boxed{Z_n \xrightarrow[q.c.]{P} \cancel{\mathbb{E}} \ \mathbb{E}Y = 2}$$

$$\mathbb{E}Y = \mathbb{E}(-\lg X) = 2$$

poiché $\mathbb{E}(-\lg X) = \int_0^1 + \lg(x)^2\, dx$

$\lg x = -z$

$x = e^{-z}$

$dx = -e^{-z}\, dz$

$$= \int_0^{+\infty} z^2 e^{-z}\, dz = \Gamma(3) = 2$$

oppure $Y = -\lg(X) \in (0, +\infty)$ q.c. e $f(x)$ è invertibile

$x = h(y) = f^{-1}(y) = e^{-y}$

$h'(y) = -e^{-y}$

$$f_Y(y) = f_X(e^{-y})\,\left|-e^{-y}\right|\,\mathbb{1}_{y \geq 0}$$

$$= -\lg(e^{-y})\, e^{-y}\,\mathbb{1}_{y \geq 0}$$

$$= y\, e^{-y}\,\mathbb{1}_{y \geq 0}$$

$$\mathbb{E}Y = \int_0^{+\infty} y^2 e^{-y}\, dy = \Gamma(3) = 2$$