Chirp

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CHIRP: linear frequency modulated signal

MAXIMUM RADAR RANGE

$$R_{max} = \sqrt[4]{\frac{E_T G^2 \lambda^2 \sigma}{(4\pi)^3 K T_0 F S_a}} \qquad \text{Con } E_T = P_p T$$

RANGE RESOLUTION

$$R_d = \frac{cT}{2}$$

CHIRP: LINEAR FREQUENCY MODULATION

$$s(t) = e^{j2\pi\left(f_p t + \frac{B}{T}\frac{t^2}{2}\right)} \, \mathrm{rect}_T(t)$$

B chirp bandwidth
T transmitted pulse length
f_p (residual) carrier frequency

- CHIRP (long pulse with phase coding): has the power properties of the long pulse and the resolution properties of the short pulse.

- Phase coding → waveform compression by means of matched filtering

CHIRP: Time domain waveform (I)

$$s(t) = e^{j2\pi(f_p t + \frac{B}{T}\cdot\frac{t^2}{2})} \mathrm{rect}_T(t)$$

- CHIRP MODULUS DEL |s(t)|:

$$|s(t)| = \begin{cases} 1 & \text{Per } |t| \le T/2 \\ 0 & \text{Per } |t| \ge T/2 \end{cases}$$

- CHIRP PHASE Φ(t)

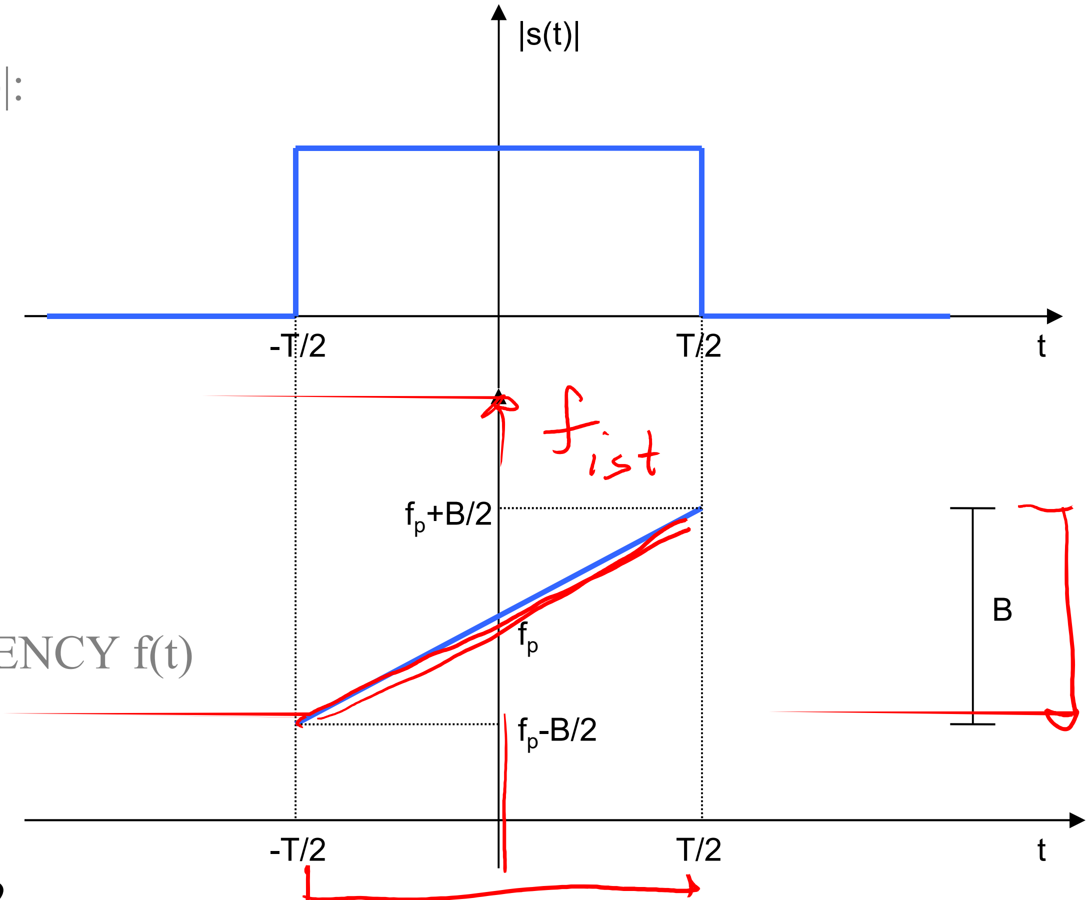
$$\Phi(t) = 2\pi(f_p t + \frac{B}{T}\cdot\frac{t^2}{2})$$

- INSTANTANEOUS FREQUENCY f(t)

$$f(t) = \frac{1}{2\pi}\cdot\frac{d\Phi(t)}{dt} = f_p + \frac{B}{T}t$$

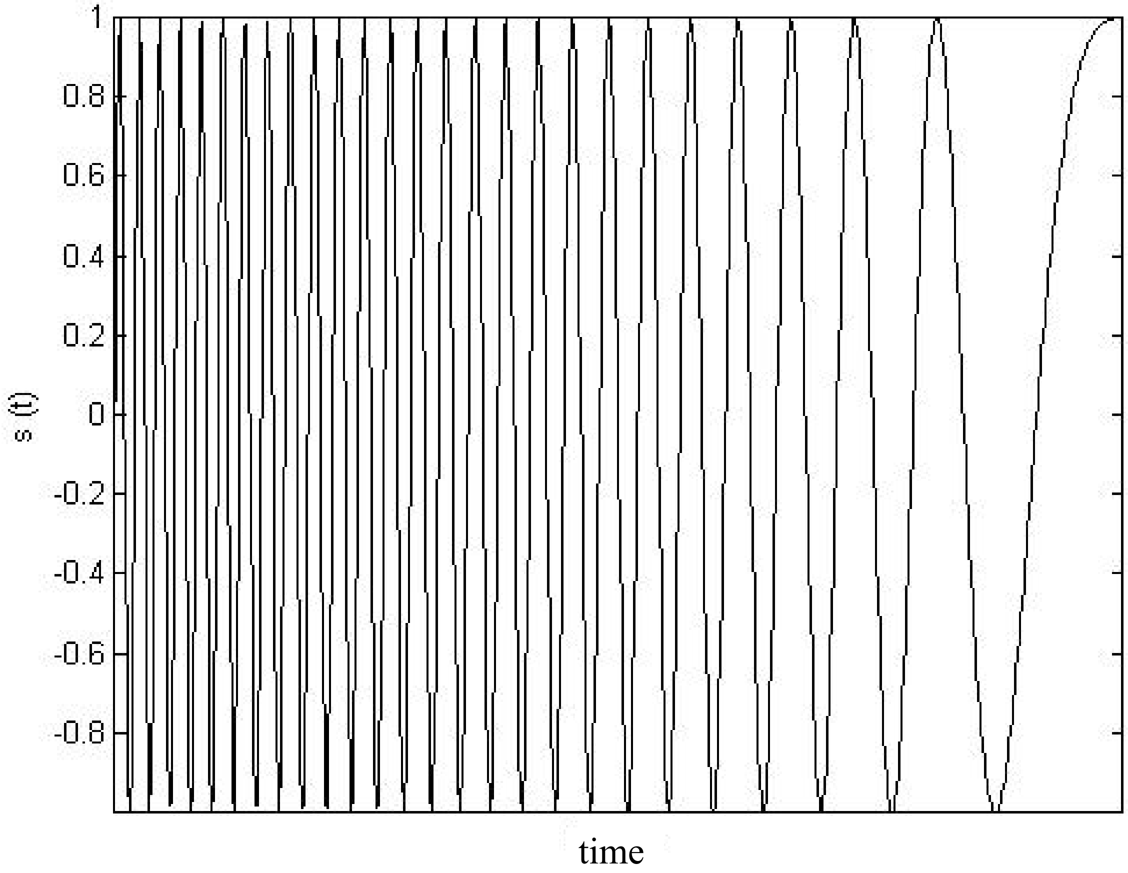
$f(-T/2)=f_p-B/2$

$f(T/2)=f_p+B/2$

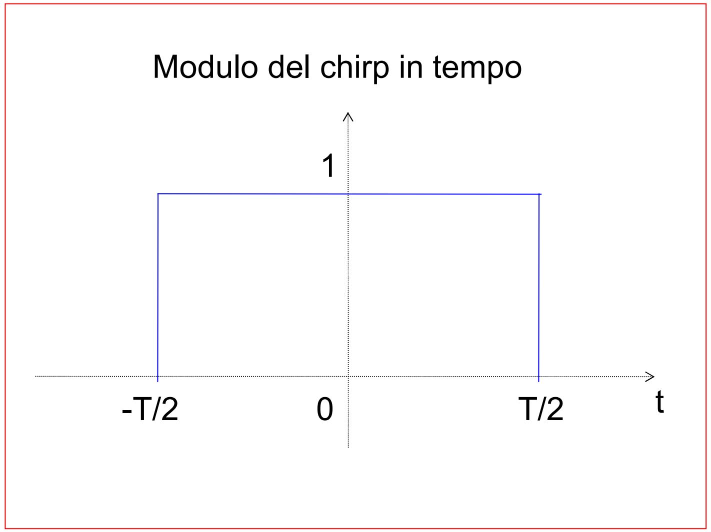


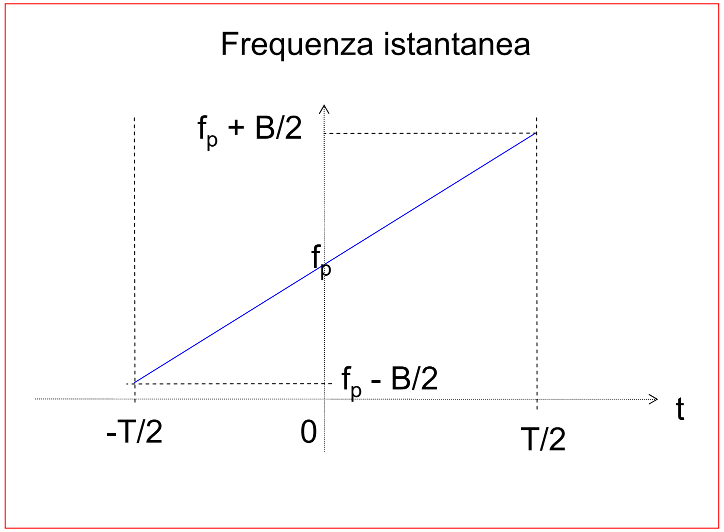
Frequency modulation

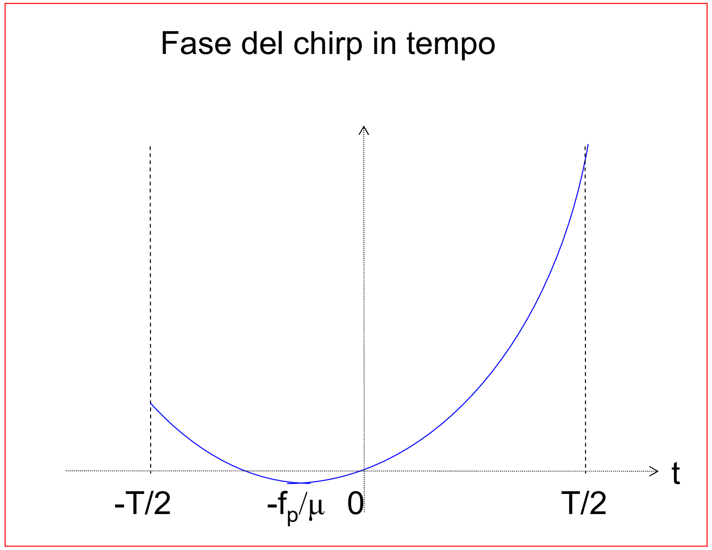
CHIRP: Time domain waveform (II)



CHIRP: Time domain waveform (III)





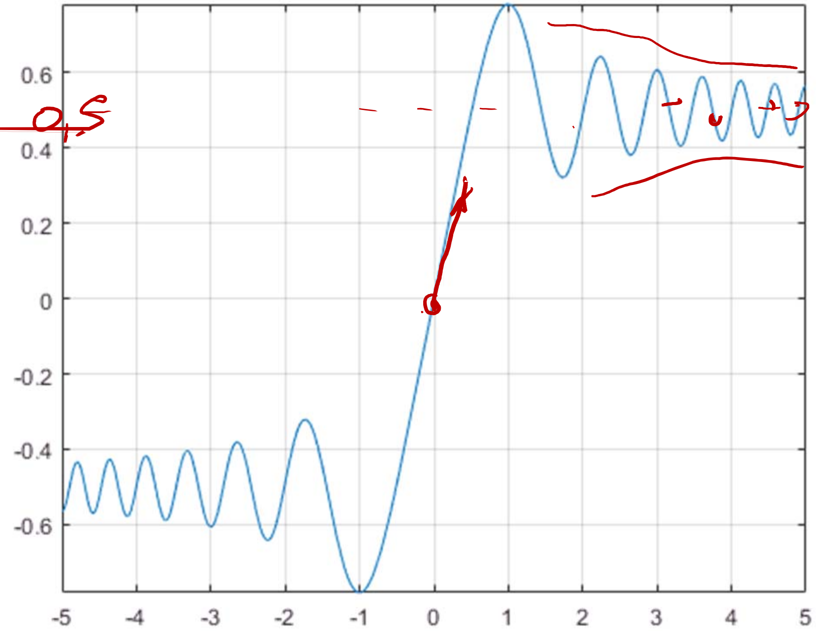


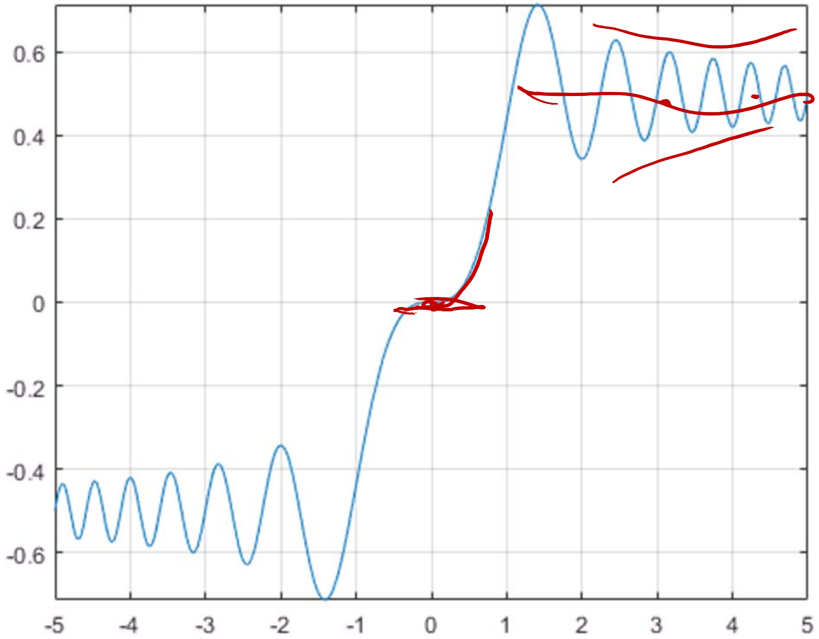
Funzioni Coseno e Seno ~~Integrale~~ Fresnel

$$C(z) = fresnelc(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right) dt$$

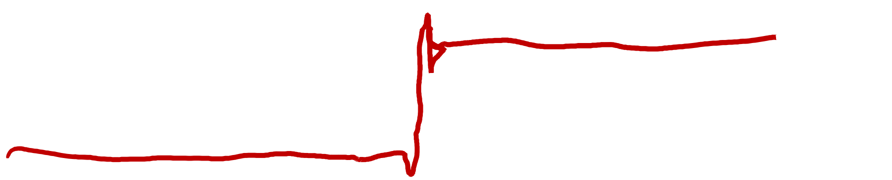
0,5

$$\boxed{\int_0^z e^{j\frac{\pi t^2}{2}} dt = \underline{C(z)} + j\,\underline{S(z)}}$$





$\frac{1}{2}$



$$S(z) = fresnels(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right) dt$$

Spettro del Chirp

$\rightarrow s(t) = e^{j\pi\frac{B}{\tau}t^2}\,\mathrm{rect}_\tau(t)$

$$S(f)=\int_{-\tau/2}^{\tau/2} e^{j\pi\frac{B}{\tau}t^2}\, e^{-j2\pi f t}\, dt =$$

$\tau = T$

- $f_{ist} = \frac{1}{\cancel{2\pi}}\,\cancel{\pi}\,\frac{B}{\tau}\,2t$

- $2\pi\left(\frac{B}{\tau}\frac{t}{2}\right)t$

$\pi\frac{B}{\tau}t^2 - 2\pi f t =$

$= \frac{\pi}{2}\left(\frac{2B}{\tau}t^2 - 4ft\right) =$

$= \frac{\pi}{2}\left(\sqrt{\frac{2B}{\tau}}\,t - \frac{2f}{\sqrt{\frac{2B}{\tau}}}\right)^2 - \frac{\pi}{\cancel{2}}\,\frac{\cancel{4}f^2}{\frac{\cancel{2}B}{\tau}}$

$$S(f) = e^{-j\pi\frac{\tau}{B}f^2}\cdot\int_{-\tau/2}^{\tau/2} e^{j\frac{\pi}{2}\Big(\underbrace{\sqrt{\frac{2B}{\tau}}\,t - \frac{2f}{\sqrt{2B/\tau}}}_{x}\Big)^2}\, dt$$

$$\sqrt{\frac{2B}{\tau}}\frac{\tau}{2}-\frac{2f}{\sqrt{2B/\tau}}$$

$$x=\sqrt{\frac{2B}{\tau}}\,t-\frac{2f}{\sqrt{2B/\tau}}$$

$$dx=\sqrt{\frac{2B}{\tau}}\,dt$$

$$S(f)=e^{-j\pi\frac{\tau}{B}f^2}\sqrt{\frac{\tau}{2B}}\cdot\int_{-\sqrt{\frac{2B}{\tau}}\frac{\tau}{2}-\frac{2f}{\sqrt{2B/\tau}}}^{\sqrt{\frac{2B}{\tau}}\frac{\tau}{2}-\frac{2f}{\sqrt{2B/\tau}}} e^{j\frac{\pi}{2}x^2}\,dx$$

$$\sqrt{2B\tau}\left(\frac{1}{2}-\frac{f}{B}\right)=x_2$$

$$-\sqrt{2B\tau}\left(\frac{1}{2}+\frac{f}{B}\right)=-x_1$$

$$\int_{-x_1}^{x_2} e^{j\frac{\pi}{2}x^2}\,dx=\int_{-x_1}^{0} e^{j\frac{\pi}{2}x^2}\,dx+\int_{0}^{x_2} e^{j\frac{\pi}{2}x^2}\,dx=$$

$$= -\int_0^{-X_1} e^{j\frac{\pi}{2}x^2}\,dx + \int_0^{X_2} e^{j\frac{\pi}{2}x^2}\,dx =$$

$$= -C(-X_1) - jS(-X_1) + C(X_2) + jS(X_2) =$$

$$= C(X_1) + jS(X_1) + C(X_2) + jS(X_2) =$$

$$= C(X_1) + C(X_2) + j\left[S(X_1) + S(X_2)\right]$$

$(1+j)$

$$\boxed{S(f) = e^{-j\pi\frac{\tau}{B}f^2}\sqrt{\frac{\tau}{2B}}\cdot\left\{\underbrace{C(x_1)+C(x_2)+j\left[S(x_1)+S(x_2)\right]}\right\}}$$

$$|S(f)|^2 = \frac{\tau}{2B}\cdot\left\{\left[C(x_1)+C(x_2)\right]^2+\left[S(x_1)+S(x_2)\right]^2\right\}$$

$$\begin{cases} x_1 = \sqrt{2B\tau}\left(\frac{1}{2}+\frac{f}{B}\right) \\ x_2 = \sqrt{2B\tau}\left(\frac{1}{2}-\frac{f}{B}\right) \end{cases}$$

$f \to \infty$ $\quad x_1 \to +\infty$

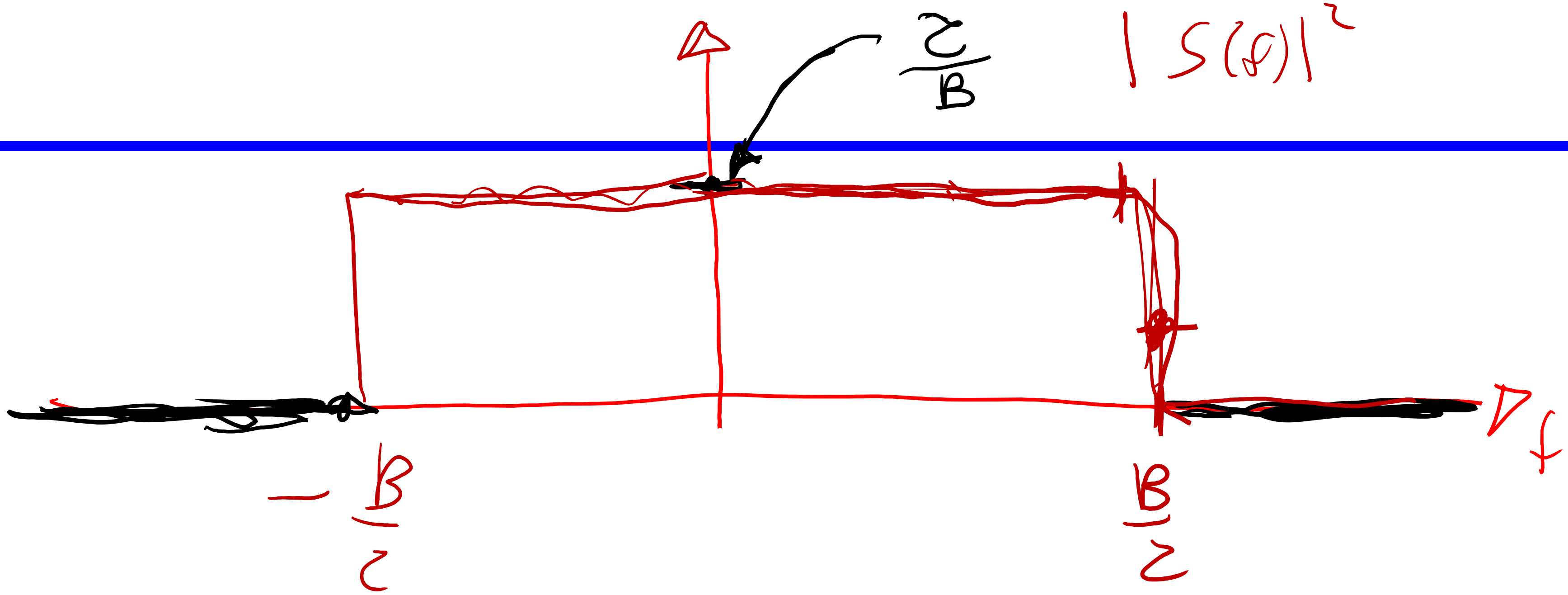
$x_2 \to -\infty$

$\Downarrow$

$C(x_1) = \frac{1}{2} = S(x_1)$

$C(x_2) = S(x_2) = -\frac{1}{2}$

$$|S(f)|^2 \to \frac{\tau}{2B}\cdot\left\{\left(\frac{1}{2}-\frac{1}{2}\right)^2+\left(\frac{1}{2}-\frac{1}{2}\right)^2\right\} = 0$$



τ/B
|S(f)|²
−B/2
B/2
f

Se $B\tau >> 1$

$$X_1 = \sqrt{2B\tau}\left(\frac{1}{2} + \frac{f}{B}\right)$$

$$X_2 = \sqrt{2B\tau}\left(\frac{1}{2} - \frac{f}{B}\right)$$

$f = 0$

$$X_1 = \sqrt{2B\tau}\cdot\frac{1}{2}$$

$$X_2 = \sqrt{2B\tau}\cdot\frac{1}{2}$$

$$|S(f)|^2 = \frac{\tau}{2B}\cdot\left\{4\,C^2\left(\sqrt{2B\tau}\cdot\frac{1}{2}\right) + 4\,S^2\left(\sqrt{2B\tau}\cdot\frac{1}{2}\right)\right\} =$$

se $B\tau >> 1$

$$C^2 \rightarrow \frac{1}{4} \qquad S^2 \rightarrow \frac{1}{4}$$

$$\boxed{\simeq \frac{\tau}{2B}\cdot 2 = \frac{\tau}{B}}$$

$\{\ \} = 2$

$f > \phi$

$X_1 = \sqrt{2B\tau}\left(\frac{1}{2} + \frac{f}{B}\right) > 0$ se $B\tau \gg 1$ anche

$X_1 \gg 1$

$C(X_1) \Rightarrow S(X_1) \Rightarrow \frac{1}{2}$

$X_2 = \sqrt{2B\tau}\left(\frac{1}{2} - \frac{f}{B}\right) \to > 0$ se $\boxed{f < \frac{B}{2}}$

$\Rightarrow$ se $B\tau \gg 1$

$X_2 \gg 1$

$C(X_2) \to \frac{1}{2}$

$S(X_2) \to \frac{1}{2}$

< 0 se $f > \frac{B}{2}$ e $B\tau \gg 1$ $X_2 \ll -1$

$C(X_2) \Rightarrow -\frac{1}{2}$

$S(X_2) \to -\frac{1}{2}$

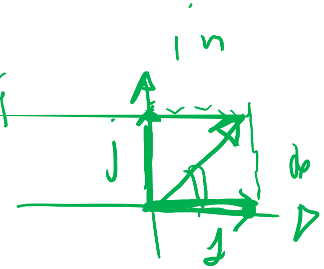
$$f = \frac{B}{2} \qquad X_2 = 0 \qquad C(X_2) = S(X_2) = 0$$

$$\frac{\tau}{2B} \cdot \left\{ \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \right\} = \frac{\tau}{4B}$$

$$B\tau \gg 1$$

$$S(f) \simeq e^{-j\pi\frac{\tau}{B}f^2}\sqrt{\frac{\tau}{2B}}\,(1+j)\,\mathrm{rect}_B(f)$$

$$\sqrt{2}\,e^{j\pi/4}$$



$$\boxed{S(f) \simeq e^{j\frac{\pi}{4}}\,e^{-j\pi\frac{\tau}{B}f^2}\sqrt{\frac{\tau}{B}}\,\mathrm{rect}_B(f)}$$

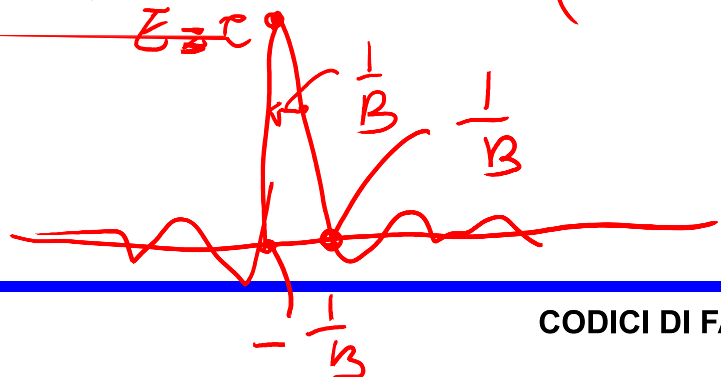
Usando approx

Autocorrelaz. del chip ?

$$R_{ss}(t) = \mathcal{F}^{-1}\left\{ |S(f)|^2 \right\}$$

$$R_{ss}(t) = \mathcal{F}^{-1}\left\{ \frac{\tau}{B}\, \mathrm{rect}_B(f) \right\} =$$

$$= \frac{\tau}{\cancel{B}}\, \cancel{B}\, \mathrm{sinc}(\pi B t) = \tau\, \mathrm{sinc}(\pi B t)$$



CHIRP: Frequency domain waveform (I)

Fourier Transform of the chirp signal:

$\mu = \frac{B}{\tau}$

$$S(f) = \frac{1}{\sqrt{2\mu}}\{[C(X_1) + C(X_2)] + j[S(X_1) + S(X_2)]\}e^{-j\frac{\pi}{\mu}f^2} = |S(f)|e^{j\Phi(f)}$$

- ✓ The compression factor BT determines the frequency domain characteristics of the chirp waveform

C(X) Fresnel cosine
S(X) Fresnel sine

$$X_1 = \sqrt{2BT}\left(\frac{1}{2} + \frac{f}{B}\right)$$

$$X_2 = \sqrt{2BT}\left(\frac{1}{2} - \frac{f}{B}\right)$$

AMPLITUDE SPECTRUM

$$|S(f)| = \frac{1}{\sqrt{2\mu}}\sqrt{[C(X_1)+C(X_2)]^2 + [S(X_1)+S(X_2)]^2}$$

For high BT values (BT>100)

$$|S(f)| \cong \frac{1}{\sqrt{2\mu}}\sqrt{2} = \frac{1}{\sqrt{\mu}} = \sqrt{\frac{T}{B}} \qquad |f| \le B/2$$

PHASE SPECTRUM

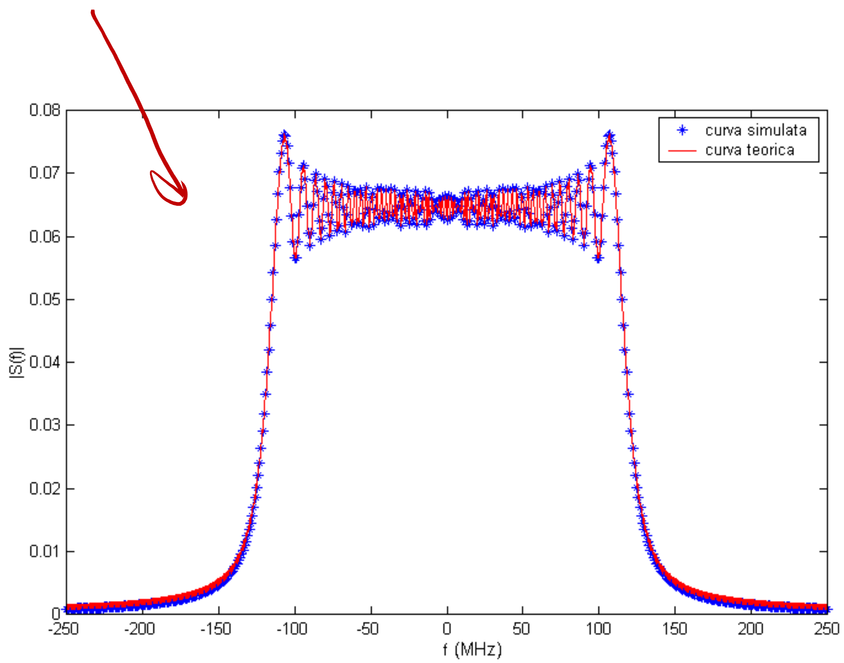
$$\Phi(f) = -\frac{\pi}{\mu}f^2 + \mathrm{atg}\left[\frac{S(X_1)+S(X_2)}{C(X_1)+C(X_2)}\right]$$

For high BT values (BT>100)

$$\Phi(f) \cong -\frac{\pi}{\mu}f^2 + \frac{\pi}{4} \qquad |f| \le B/2$$

$$\boxed{S(f) = \sqrt{\frac{T}{B}}e^{-j\left[\pi\frac{T}{B}f^2 - \frac{\pi}{4}\right]}\mathrm{rect}_B(f)}$$

CHIRP: Frequency domain waveform (II)



Chirp amplitude spectrum for B=240MHz, T=1μs

Chirp phase spectrum for B=240MHz, T=1μs