Gestione della soglia di Rivelazione

Probabilità di falso allarme – soglia fissa

Sotto l'ipotesi H_0 (a=0) ho un Falso Allarme se

$$z = \left|\tilde{z}\right| = \left|d_f(t)\right| > T$$

Poiché $d_f(t)$ è Gaussiana a valor medio nullo e varianza σ_d^2 ($\sigma_d^2 = \sigma_n^2$ se disturbo = solo rumore termico):

$$p(\tilde{z}\,|\,H_0) = \frac{1}{\pi\sigma_d^2}\exp\left\{-\frac{1}{\sigma_d^2}\left|\tilde{z}\right|^2\right\} \Rightarrow p_z(z|\,H_0) = \frac{2z}{\sigma_d^2}e^{-\frac{z^2}{\sigma_d^2}}$$

$$\boxed{P_{fa} = \Pr ob\{z > T\,|\,H_0\} = \int_T^\infty p_z(z|\,H_0)\,dz = e^{-\frac{T^2}{\sigma_d^2}}}$$

Scelta della soglia fissa

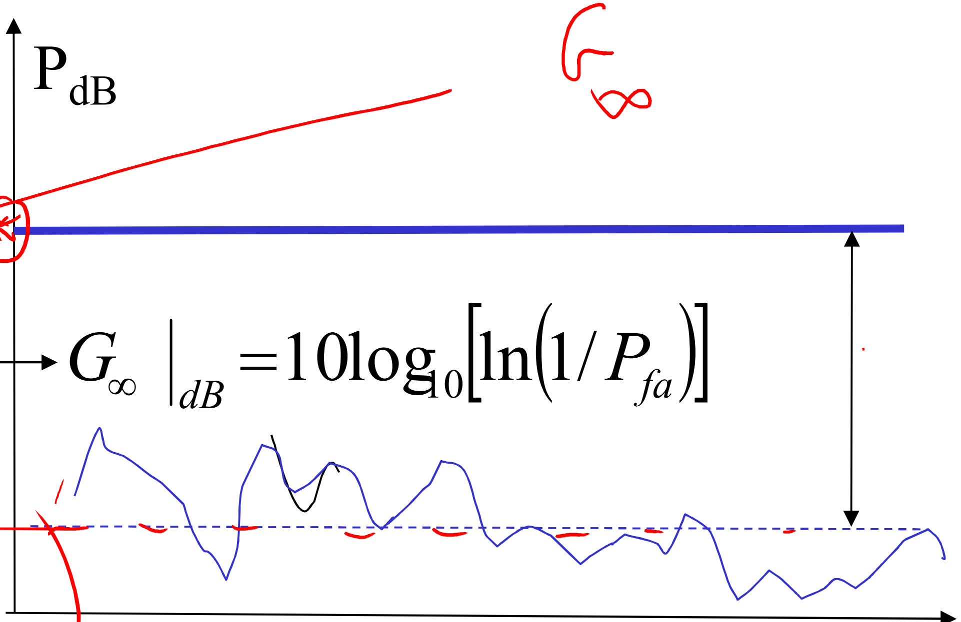
$$P_{fa} = e^{-\frac{T^2}{\sigma_d^2}} \longrightarrow \frac{T^2}{\sigma_d^2} = -\ln P_{fa}$$

$* \; T\big|_{dB} = \sigma_d\big|_{dB} + G_\infty$

$$\boxed{T = \sigma_d \cdot \sqrt{-\ln P_{fa}}}$$

$$\boxed{T^2 = \sigma_d^2 \cdot \ln\left(1/P_{fa}\right)}$$

G_∞



$$G_\infty\big|_{dB} = 10\log_{10}\left[\ln\left(1/P_{fa}\right)\right]$$

$$\sigma_d^2\big|_{dB}$$

Livello di potenza noto

Scelta della soglia – stima del disturbo

$$\boxed{T^2 = \sigma_d^2 \cdot \ln\left(1/P_{fa}\right)} \longrightarrow G_\infty \big|_{dB}$$

$$\longrightarrow \sigma_d^2 \big|_{dB}$$

$F\,KT \cdot B$

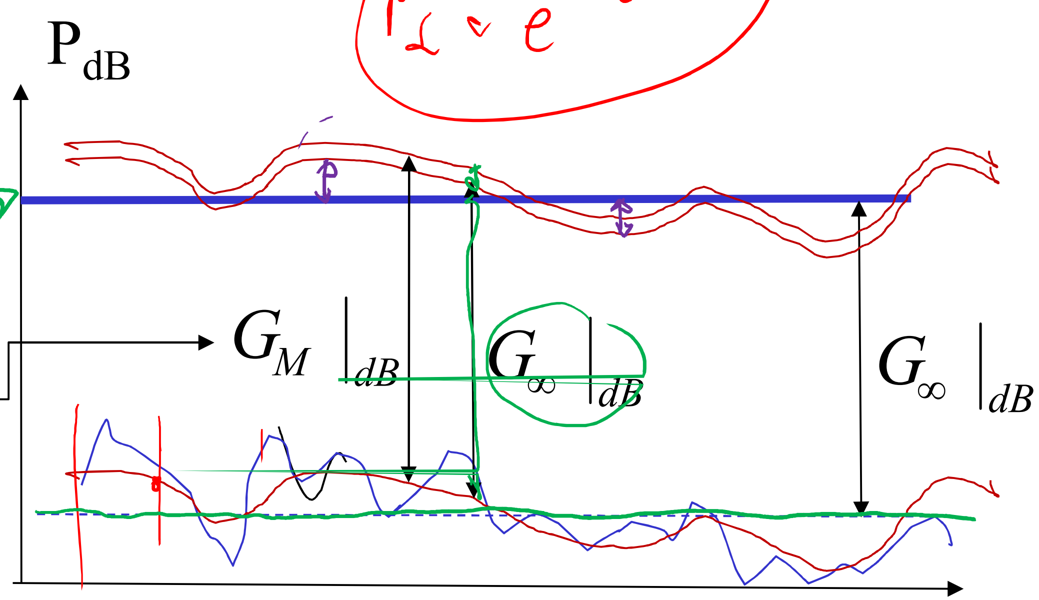
N_0

$P_a = e^{-G}$

Livello di potenza noto

$$\boxed{T^2 = \hat{\sigma}_d^2 \cdot G_M} \longrightarrow G_M \big|_{dB}$$

$$\longrightarrow \hat{\sigma}_d^2 \big|_{dB}$$



Livello di potenza stimato

$P_a = e^{-G}$ $\quad \ln P_a = -G$

$G = -\ln P_a$

Incremento di P_{fa} per stima disturbo

$$P_{fa}=\frac{1}{2}e^{-\frac{G\cdot(\sigma_d^2+\Delta)}{\sigma_d^2}}+\frac{1}{2}e^{-\frac{G\cdot(\sigma_d^2-\Delta)}{\sigma_d^2}}=\frac{1}{2}e^{-G-\frac{G\Delta}{\sigma_d^2}}+\frac{1}{2}e^{-G+\frac{G\Delta}{\sigma_d^2}}=$$

$$=e^{-G}\frac{1}{2}\left[e^{-\frac{G\Delta}{\sigma_d^2}}+e^{\frac{G\Delta}{\sigma_d^2}}\right]=e^{-G}\frac{1}{2}\left[1-\cancel{\frac{G\Delta}{\sigma_d^2}}+\frac{1}{2}\left(\frac{G\Delta}{\sigma_d^2}\right)^2+1+\cancel{\frac{G\Delta}{\sigma_d^2}}+\frac{1}{2}\left(\frac{G\Delta}{\sigma_d^2}\right)^2\right]=$$

$$=e^{-G}\left[1+\frac{1}{2}\left(\frac{G\Delta}{\sigma_d^2}\right)^2\right] > 1$$

~~con prob.~~ $\frac{1}{2}$ $\quad \hat{\sigma}_d^2=\sigma_d^2+\Delta \rightarrow T^2=(\sigma_d^2+\Delta)\cdot G$

$P_{fa}=e^{-\frac{(\sigma_d^2+\Delta)G}{\sigma_d^2}}$ nel caso di stime per eccesso

P_{fa} media = ?

~~un~~ valore noto

$P_{fa}=e^{-\frac{\sigma_d^2\cdot G}{\sigma_d^2}}=e^{-G}$

$\frac{1}{2}$ $\quad \hat{\sigma}_d^2=\sigma_d^2-\Delta \rightarrow T^2=(\sigma_d^2-\Delta)\cdot G$

P_{fa} per stima per difetto

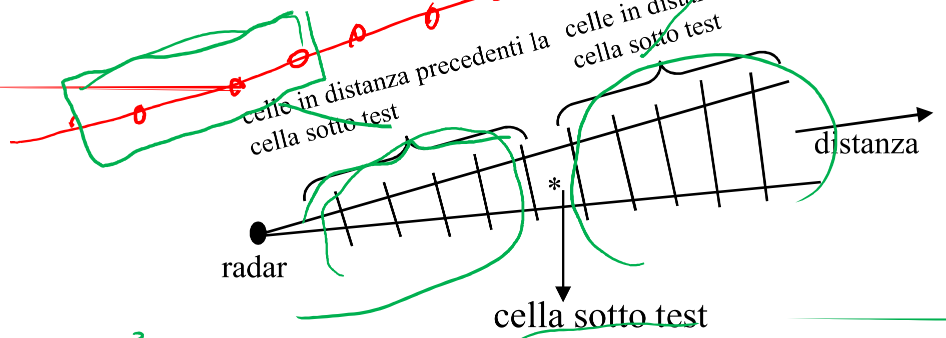
$P_{fa}=e^{-\frac{(\sigma_d^2-\Delta)G}{\sigma_d^2}}$

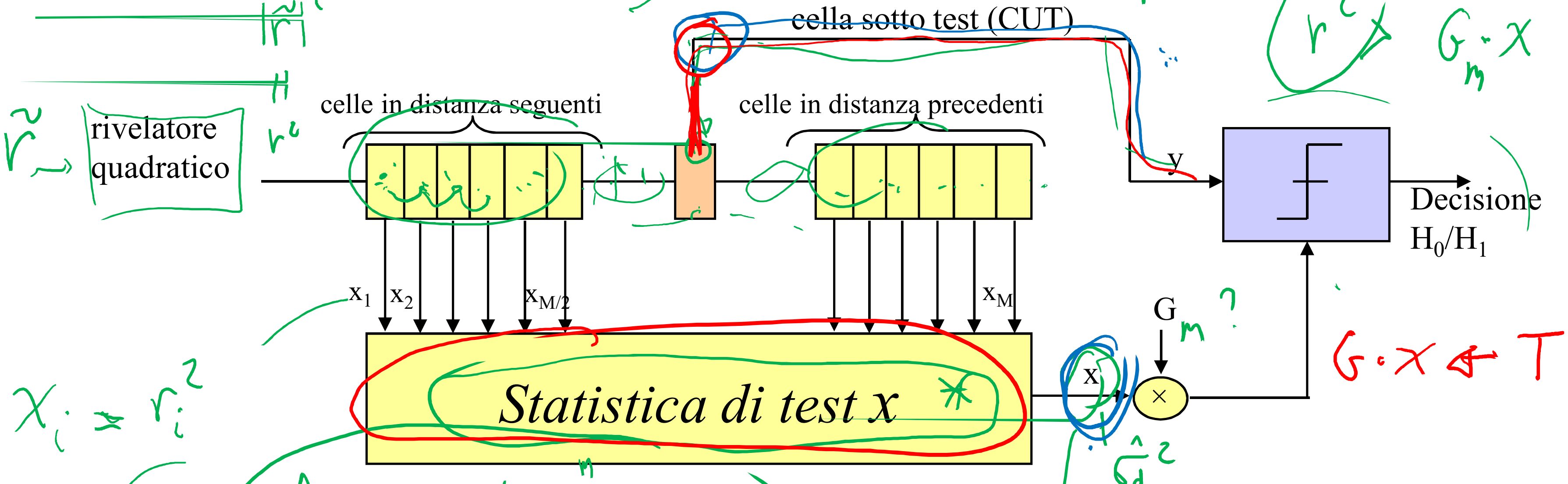
Autogate (I)

CFAR

(CA)-CFAR

CELL AVERAGE





$G \cdot x \gtrless T$

$x_i = r_i^2$

$x = \hat{\sigma}_d^2 = \frac{1}{M}\sum_{i=1}^{M} x_i$

G = Guadagno che determina la P_{fa}

PDF statistica di test (I)

$x_i = r_i^2 = |\tilde{r}_i|^2$

Statistica di test $x = \frac{1}{M}\sum_{i=1}^{M} x_i = \frac{1}{M}x'$

$y = \alpha x \qquad p_y(y) = \frac{1}{\alpha} p_x\left(\frac{y}{\alpha}\right)$

$x' = \sum_{i=1}^{M} x_i$

- Ipotesi: disturbo gaussiano

$$p_y(y/H_0) = \frac{1}{\sigma^2} e^{-\frac{y}{\sigma^2}} \quad y \geq 0$$

$$p_{x_i}(x_i) = \frac{1}{\sigma^2} e^{-\frac{x_i}{\sigma^2}} \quad x_i \geq 0, i = 1 \ldots M$$

⇒ M variabili aleatorie indipendenti identicamente distribuite

Per l'ipotesi di indipendenza statistica:
densità di probabilità = convoluzione delle singole densità

⇒ $p_{x'}(x') = p_{x_1}(x') * p_{x_2}(x') \ldots * p_{x_M}(x')$

Funzione caratteristica:

$$C_{x'}(u) = C_{x_1}(u) \cdot C_{x_2}(u) \cdot \ldots \cdot C_{x_M}(u) = \left[C_{x_i}(u)\right]^M = \left[\frac{1/\sigma^2}{1/\sigma^2 - ju}\right]^M$$

⇒ antitrasformando

$$p_{x'}(x') = \frac{1}{(M-1)!}\frac{1}{\sigma^2}\left(\frac{x'}{\sigma^2}\right)^{M-1} e^{-\frac{x'}{\sigma^2}} \quad x' \geq 0$$

$$x = \frac{x'}{M} \Rightarrow p_x(x) = M p_{x'}(Mx)$$

$$C_{x'}(u) = \int_{-\infty}^{+\infty} \cancel{p_{x'}(x')}\, e^{-j2\pi u x'}\, dx'$$

$$\|$$

$$C_{x_1}(u)\cdot C_{x_2}(u)\cdot C_{x_3}(u)\cdot \ldots\ldots \quad C_{x_M}(u) =$$

$$= \boxed{\frac{1}{(1+j2\pi\sigma^2 u)^M}}$$

$$e^{-\left(\frac{1}{\sigma^2}+j2\pi u\right)x_1}$$

$$C_{x_1}(u) = \int_0^{+\infty} \frac{1}{\sigma^2}\, e^{-\frac{x_1}{\sigma^2}}\, e^{-j2\pi u x_1}\, dx_1 = \frac{1}{\sigma^2}\,\frac{\cancel{+}1}{\cancel{+}\left(\frac{1}{\sigma^2}+j2\pi u\right)} = \frac{1}{1+j2\pi u\sigma^2}$$

Radiotecnica e Radiolocalizzazione

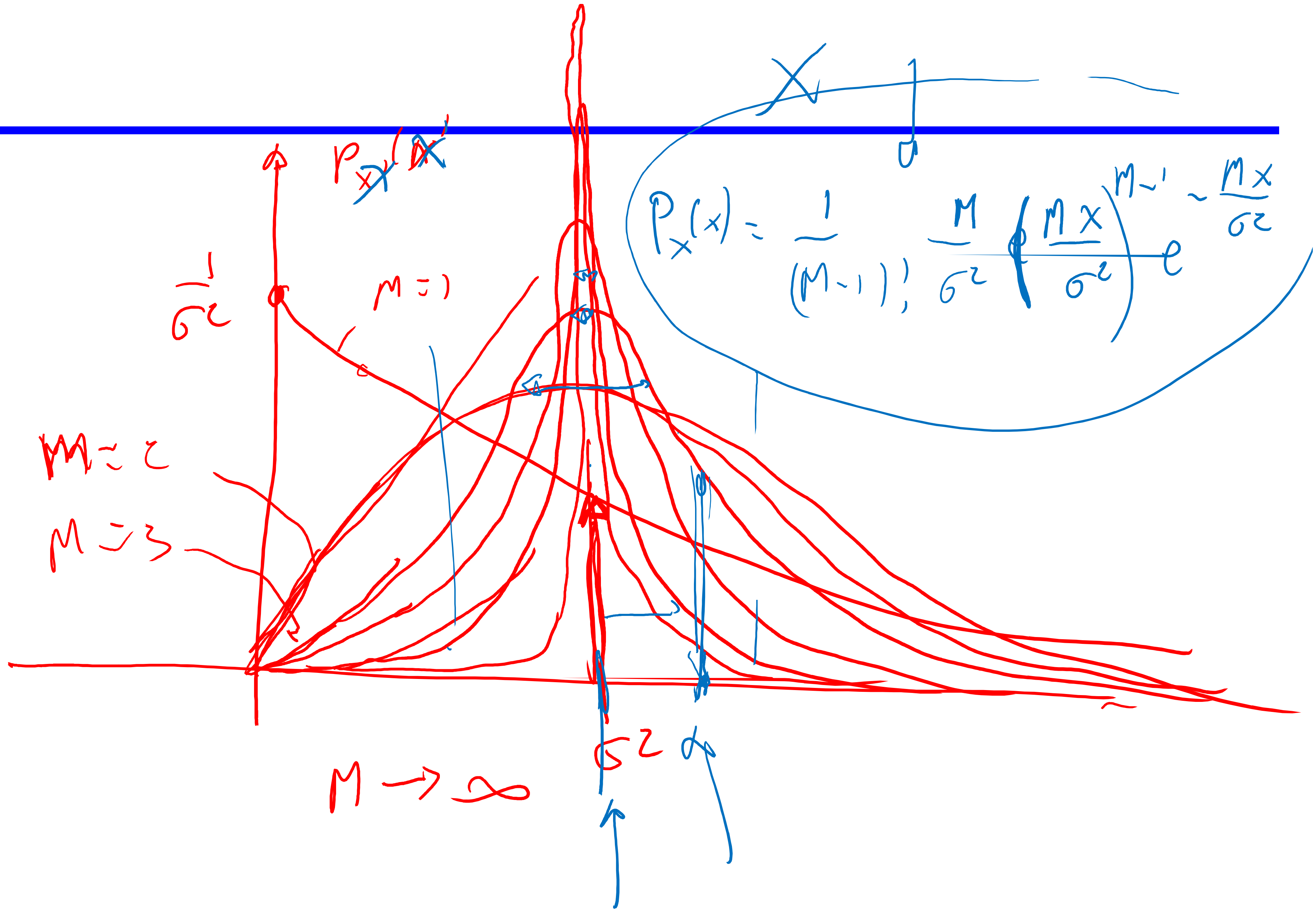
$$x' = \sum_{i=1}^{M} x_i = \sum_{i=1}^{M} |\tilde{r}_i|^2 = \sum_{i=1}^{M} (r_{iI}^2 + r_{iQ}^2)$$

somma dei quadrati di $2M$ variabili aleatorie gaussiane indip. con v.m $= 0$ ~~e~~ stessa varianza σ^2

$$\boxed{p_{x'}(x') = \frac{1}{(M-1)!} \cancel{x} \frac{1}{\sigma^2} \left(\frac{x'}{\sigma^2}\right)^{M-1} e^{-\frac{x'}{\sigma^2}} \quad x' > 0}$$

DDP GAMMA

$M = 1$ $\quad \frac{1}{\sigma^2} e^{-\frac{x'}{\sigma^2}}$


$p_x(x) = \frac{1}{(M-1)!}\frac{M}{\sigma^2}\left(\frac{Mx}{\sigma^2}\right)^{M-1} e^{-\frac{Mx}{\sigma^2}}$
$p_x(x)$
$\frac{1}{\sigma^2}$
$M=1$
$M=2$
$M=3$
$M \to \infty$
σ^2
α

DDP somma variabili esponenziali (I)

$$p_{x_i}(x_i)=\frac{1}{\sigma^2}e^{-\frac{x_i}{\sigma^2}}u(x_i)$$

$$p(x',2)=p_{x'}(x')*p_{x'}(x')=\frac{1}{\sigma^2}e^{-\frac{x'}{\sigma^2}}u(x')*\frac{1}{\sigma^2}e^{-\frac{x'}{\sigma^2}}u(x')=\int_0^\infty\frac{1}{\sigma^2}e^{-\frac{t}{\sigma^2}}\cancel{u(t)}\frac{1}{\sigma^2}e^{-\frac{x'-t}{\sigma^2}}u(x'-t)dt=$$

= 1 per $x'>t$

$t<x'$

$$=\left[\int_0^{x'}\frac{1}{\sigma^2}e^{-\frac{t}{\sigma^2}}\frac{1}{\sigma^2}e^{-\frac{x'-t}{\sigma^2}}dt\right]u(x')=\left[\frac{1}{(\sigma^2)^2}e^{-\frac{x'}{\sigma^2}}\int_0^{x'}dt\right]u(x')=\boxed{\frac{x'}{(\sigma^2)^2}e^{-\frac{x'}{\sigma^2}}u(x')}$$

$$p(x',3)=p(x',2)*p_{x'}(x')=\frac{x'}{(\sigma^2)^2}e^{-\frac{x'}{\sigma^2}}u(x')*\frac{1}{\sigma^2}e^{-\frac{x'}{\sigma^2}}u(x)=\int_0^\infty\frac{t}{(\sigma^2)^2}e^{-\frac{t}{\sigma^2}}u(t)\frac{1}{\sigma^2}e^{-\frac{x'-t}{\sigma^2}}u(x'-t)dt=$$

$$=\left[\int_0^{x'}\frac{t}{(\sigma^2)^2}e^{-\frac{t}{\sigma^2}}\frac{1}{\sigma^2}e^{-\frac{x'-t}{\sigma^2}}dt\right]u(x')=\left[\frac{1}{(\sigma^2)^3}e^{-\frac{x'}{\sigma^2}}\int_0^{x'}t\,dt\right]u(x')=\boxed{\frac{x'^2}{2(\sigma^2)^3}e^{-\frac{x'}{\sigma^2}}u(x')}$$

$$p(x',4)=p(x',3)*p_{x'}(x')=\frac{x'^2}{2(\sigma^2)^3}e^{-\frac{x'}{\sigma^2}}u(x')*\frac{1}{\sigma^2}e^{-\frac{x'}{\sigma^2}}u(x')=\int_0^\infty\frac{t^2}{2(\sigma^2)^3}e^{-\frac{t}{\sigma^2}}u(t)\frac{1}{\sigma^2}e^{-\frac{x'-t}{\sigma^2}}u(x'-t)dt=$$

$$=\left[\int_0^{x'}\frac{t^2}{2(\sigma^2)^3}e^{-\frac{t}{\sigma^2}}\frac{1}{\sigma^2}e^{-\frac{x'-t}{\sigma^2}}dt\right]u(x')=\left[\frac{1}{2(\sigma^2)^4}e^{-\frac{x'}{\sigma^2}}\int_0^{x'}t^2\,dt\right]u(x')=\boxed{\frac{x'^3}{3\cdot 2(\sigma^2)^4}e^{-\frac{x'}{\sigma^2}}u(x')}$$

DDP somma variabili esponenziali (II)

$$p(x',5)=p(x',4)*p_{x'}(x')=\frac{x'^3}{3\cdot 2(\sigma^2)^4}e^{-\frac{x'}{\sigma^2}}u(x')*\frac{1}{\sigma^2}e^{-\frac{x'}{\sigma^2}}u(x')=\int_0^{\infty}\frac{t^3}{3\cdot 2(\sigma^2)^4}e^{-\frac{t}{\sigma^2}}u(t)\frac{1}{\sigma^2}e^{-\frac{x'-t}{\sigma^2}}u(x'-t)dt=$$

$$=\left[\int_0^{x'}\frac{t^3}{3\cdot 2(\sigma^2)^4}e^{-\frac{t}{\sigma^2}}\frac{1}{\sigma^2}e^{-\frac{x'-t}{\sigma^2}}dt\right]u(x')=\left[\frac{1}{3\cdot 2(\sigma^2)^5}e^{-\frac{x'}{\sigma^2}}\int_0^{x'}t^3\,dt\right]u(x')=\boxed{\frac{x'^4}{4\cdot 3\cdot 2(\sigma^2)^5}e^{-\frac{x'}{\sigma^2}}u(x')}$$

$$\boxed{p(x',M)=\frac{1}{(M-1)!}\frac{x'^{M-1}}{(\sigma^2)^M}e^{-\frac{x'}{\sigma^2}}u(x')}$$

Gamma

$$\boxed{p(x',M+1)}=p(x',M)*p_{x'}(x')=\frac{1}{(M-1)!}\frac{x'^{M-1}}{(\sigma^2)^N}e^{-\frac{x'}{\sigma^2}}u(x')*\frac{1}{\sigma^2}e^{-\frac{x'}{\sigma^2}}u(x')=$$

$$=\int_0^{\infty}\frac{1}{(M-1)!}\frac{t^{M-1}}{(\sigma^2)^N}e^{-\frac{t}{\sigma^2}}u(t)\frac{1}{\sigma^2}e^{-\frac{x'-t}{\sigma^2}}u(x'-t)dt=\left[\int_0^{x'}\frac{1}{(M-1)!}\frac{t^{M-1}}{(\sigma^2)^M}e^{-\frac{t}{\sigma^2}}\frac{1}{\sigma^2}e^{-\frac{x'-t}{\sigma^2}}dt\right]u(x')=$$

$$=\left[\frac{1}{(M-1)!}\frac{1}{(\sigma^2)^{M+1}}e^{-\frac{x'}{\sigma^2}}\int_0^{x'}t^{M-1}\,dt\right]u(x')=\boxed{\frac{1}{M!}\frac{x'^M}{(\sigma^2)^{M+1}}e^{-\frac{x'}{\sigma^2}}u(x')}$$

DDP somma variabili esponenziali (III)

$$\boxed{\int_0^\infty p(x',M)\,dx' = \int_0^\infty \frac{1}{(M-1)!}\,\frac{x'^{M-1}}{(\sigma^2)^M}\,e^{-\frac{x'}{\sigma^2}}\,u(x')\,dx' = 1}$$

$$m_{x'}^k = \int_0^\infty x'^k\,p(x',M)\,dx' = \int_0^\infty x'^k\,\frac{1}{(M-1)!}\,\frac{x'^{M-1}}{(\sigma^2)^M}\,e^{-\frac{x'}{\sigma^2}}\,u(x')\,dx' =$$

$$= \frac{(M+k-1)!}{(M-1)!}(\sigma^2)^k \boxed{\int_0^\infty \frac{1}{(M+k-1)!}\,\frac{x'^{M+k-1}}{(\sigma^2)^{M+k}}\,e^{-\frac{x'}{\sigma^2}}\,u(x')\,dx' =} \; = 1$$

$$= \frac{(M+k-1)!}{(M-1)!}(\sigma^2)^k = (M+k-1)\cdot(M+k-2)\cdot\ldots\cdot M\cdot(\sigma^2)^k$$

$$m_{x'}^1 = M\cdot\sigma^2 \qquad\qquad m_{x'}^2 = (M+1)\cdot M\cdot(\sigma^2)^2$$

$$\mathrm{var} = m_{x'}^2 - (m_{x'}^1)^2 = (M+1)\cdot M\cdot(\sigma^2)^2 - M^2\cdot(\sigma^2)^2 = \boxed{M\cdot(\sigma^2)^2}$$

$$C = \frac{\mathrm{var}}{(m_{x'}^1)^2} = \frac{m_{x'}^2 - (m_{x'}^1)^2}{(m_{x'}^1)^2} = \frac{M\cdot(\sigma^2)^2}{M^2\cdot(\sigma^2)^2} = \frac{1}{M}$$

DDP media variabili esponenziali (I)

$$x = \frac{1}{M} x' \qquad p(x, M) = M \cdot p(M\,x, M)$$

$$p(x, M) = \frac{M}{(M-1)!} \frac{(M\,x)^{M-1}}{(\sigma^2)^M} e^{-\frac{M\,x}{\sigma^2}} u(x) = \frac{1}{(M-1)!} \left(\frac{M}{\sigma^2}\right)^M x^{M-1}\, e^{-\frac{M\,x}{\sigma^2}} u(x)$$

$$m_x^k = E\left\{\left(\frac{1}{M} x'\right)^k\right\} = \frac{1}{M^k} m_{x'}^k = \frac{(M+k-1)!}{M^k (M-1)!} (\sigma^2)^k = \frac{(M+k-1)\cdot(M+k-2)\cdot\ldots(M+1)\cdot M}{M^k} \cdot (\sigma^2)^k =$$

$$= (1 + \frac{k-1}{M}) \cdot (1 + \frac{k-2}{M}) \cdot \ldots \cdot (1 + \frac{1}{M}) \cdot 1 \cdot (\sigma^2)^k$$

$$m_x^1 = \sigma^2 \qquad m_x^2 = (1 + \frac{1}{M}) \cdot (\sigma^2)^2$$

$$\mathrm{var} = m_x^2 - (m_x^1)^2 = (1 + \frac{1}{M}) \cdot (\sigma^2)^2 - (\sigma^2)^2 = \frac{1}{M} (\sigma^2)^2$$

$$C = \frac{\mathrm{var}}{(m_x^1)^2} = \frac{m_x^2 - (m_x^1)^2}{(m_x^1)^2} = \frac{1}{M} \frac{(\sigma^2)^2}{(\sigma^2)^2} = \frac{1}{M}$$

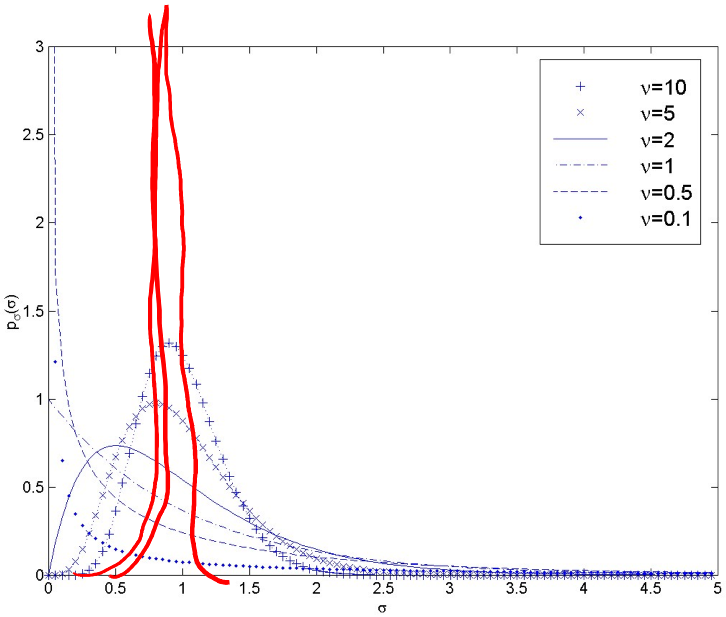
La DDP gamma

Simple solution is given by the gamma PDF:

$$p(t)=\left(\frac{\nu}{\sigma^2}\right)^{\nu}\frac{t^{\nu-1}}{\Gamma(\nu)}\exp\left[-\frac{\nu}{\sigma^2}\,t\right]$$

where n is an order parameter, with moments given by

$$\left\langle t^n\right\rangle=\left(\frac{\sigma^2}{\nu}\right)^{n}\frac{\Gamma(n+\nu)}{\Gamma(\nu)}$$



The square of the contrast or CV, i.e. the normalised variance, is

$$\boxed{\operatorname{var} t/(\sigma^2)^2=1/\nu}$$

Autogate (II)

$* \; P_{fa} = \int_T^{\infty} p(r/H_0)\, dr$

Probabilità di falso allarme:

$$P_{fa} = \int_0^{\infty} \left[\int_{G\cdot x}^{\infty} p_y(y/H_0)dy \right] p_x(x)\, dx$$

$$P_{fa}(x) = \int_{Gx}^{\infty} p(y/H_0)\, dy$$

P_{fa}/x

DDP della cella sotto test

DDP della statistica di test

Probabilità di rivelazione:

$$P_d = \int_0^{\infty} \left[\int_{G\cdot x}^{\infty} p_y(y/H_1)dy \right] p_x(x)\, dx$$

$$P_{fa} = \int P_{fa}(x)\, p_x(x)\, dx$$

$$\int p(z/x) \cdot p_x(x)\, dx$$

CA-CFAR (II)

$$\left[-e^{-\frac{y}{\sigma^2}}\right]_{Gx}^{\infty} = e^{-\frac{Gx}{\sigma^2}}$$

$$P_{fa} = \int_0^{\infty}\left[\int_{Gx}^{\infty}\frac{1}{\sigma^2}e^{-\frac{y}{\sigma^2}}dy\right]p_x(x)dx = \int_0^{\infty} e^{-\frac{Gx}{\sigma^2}}p_x(x)dx = \int_0^{\infty} e^{-\frac{Gx}{\sigma^2}}\frac{M}{(M-1)!}\frac{1}{\sigma^2}\left(\frac{Mx}{\sigma^2}\right)^{M-1}e^{-\frac{Mx}{\sigma^2}}dx =$$

$$= \int_0^{\infty}\frac{M}{(M-1)!}\frac{1}{\sigma^2}\left(\frac{Mx}{\sigma^2}\right)^{M-1}e^{-\frac{x}{\sigma^2}(M+G)}dx = \left(\frac{M}{M+G}\right)^{M}\int_0^{\infty}\underbrace{\frac{1}{(M-1)!}\frac{1}{\sigma^2}\left(\frac{z}{\sigma^2}\right)^{M-1}e^{-\frac{z}{\sigma^2}}}_{\text{Integrale di una densità di probabilità}}dz = \boxed{\left(1+\frac{G}{M}\right)^{-M}}$$

$z = x(M+G)$

$z = (M+G)x$

Fissato il tasso di falsi allarmi desiderato e fissato il valore di M resta fissato il valore da dare al guadagno G.

$$\frac{M^M}{(M+G)^M} = \left(\frac{M}{M+G}\right)^M \qquad \left(\frac{1}{1+\frac{G}{M}}\right)^M = \frac{1}{\left(1+\frac{G}{M}\right)^M}$$

Probabilità di falso allarme dipende solo da G e da M ed è indipendente dal livello del disturbo ⇒ **CFAR: Constant False Alarm Rate** indipendente da variazioni spaziali o temporali del livello del disturbo (la soglia si adatta a variazioni spaziali e/o temporali di σ^2).

$$\boxed{P_{fa}(M) = \left(1+\frac{G}{M}\right)^{-M}} \qquad P_{fa}^{-1/M} = 1+\frac{G}{M}$$

$$G_M = M\cdot\left[P_{fa}^{-1/M} - 1\right]$$

Al crescere di M G_M ~~decresce~~

$$\lim_{M\to\infty} P_{fa}(M) = \lim_{M\to\infty}\left(1+\frac{G}{M}\right)^{-M} = e^{-G}$$

$$P_{fa} = e^{-\frac{G\cdot\cancel{\sigma^2}}{\cancel{\sigma^2}}}$$

CA-CFAR (III)

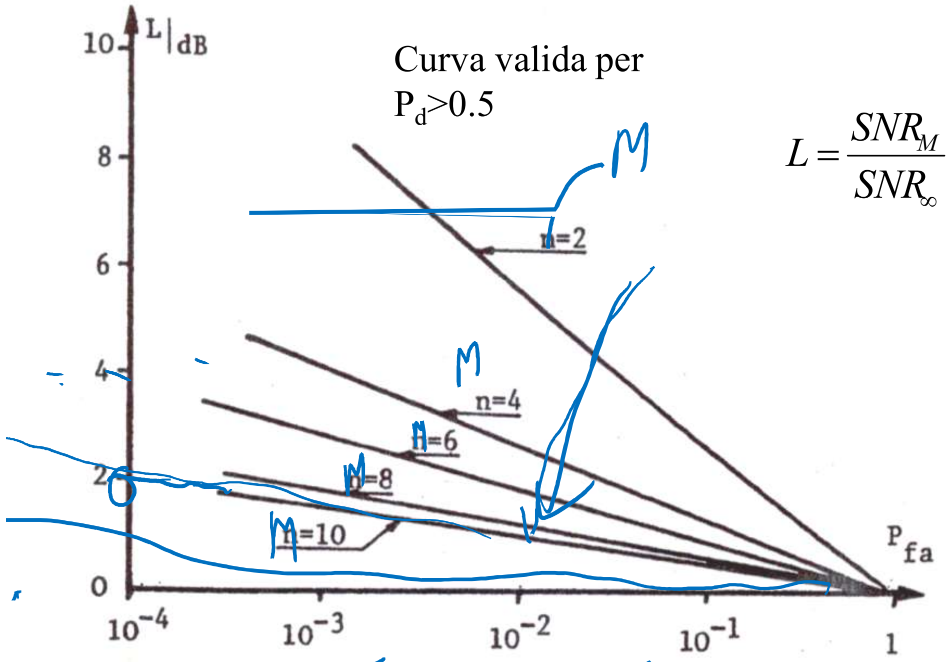
$P_{fa} = 10^{-6}$

$P_d = 0{,}9$

- Probabilità di rivelazione: $$P_d = \int_0^{\infty}\left[\int_{Gx}^{\infty} p_y(y/H_1)dy\right]p_x(x)dx$$

$\frac{S}{N} = 13\,dB$

- soglia fluttuante $\rightarrow$ perdite in sensibilità (tutto va come se ci fosse una soglia fissa più un rumore additivo dovuto alla fluttuazione della stima)



$$L = \frac{SNR_M}{SNR_\infty}$$

Perdite in sensibilità: rapporto tra SNR richiesto in presenza di soglia stimata adattivamente da M campioni (SNR_M) e SNR richiesto in presenza di soglia fissa (SNR_∞).

Fissato il livello di falso allarme le perdite sono tanto minori quanto maggiore è M (M=∞ equivale a disturbo noto).

Loss 4dB → 17dB