

0,4 ns

10  $\mu$ s

200 ms

# Proprietà della Funzione di Ambiguità

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1  $\Delta$ , 2  $\Delta$

2D-CAF



# Funzione di ambiguità (VII)

$$|\chi(\tau, \nu)| = \left| \int_{-\infty}^{\infty} s_0(t) s_0^*(t + \tau) e^{j2\pi\nu t} dt \right|$$

– **Proprietà della Funzione di Ambiguità**

(assumendo che l'energia di  $s_0(t)$  sia unitaria)

1. Massimo in (0,0)  $\rightarrow E$

$$|\chi(\tau, \nu)| \leq |\chi(0,0)| = 1 - E$$

2. Volume Costante

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi(\tau, \nu)|^2 d\tau d\nu = 1$$

3. Simmetria rispetto all'origine

$$|\chi(-\tau, -\nu)| = |\chi(\tau, \nu)|$$

4. Modulazione lineare di frequenza

$$s_0(t) \Leftrightarrow |\chi(\tau, \nu)|$$

$$s_0(t) e^{j\pi k t^2} \Leftrightarrow |\chi(\tau, \nu - k\tau)|$$

$$\phi = \pi k t^2$$

$$f_{ist} = \frac{1}{2\pi} \frac{d\phi(t)}{dt} =$$

$$= \frac{1}{2} k 2t$$

$$= kt$$

# Funzione di ambiguità (VIII)

- **Proprietà #1 della Funzione di Ambiguità:** Massimo in (0,0)

Dalla disuguaglianza di Schwartz;

$$|\int f \cdot g dt|^2 \leq \int |f|^2 dt \cdot \int |g|^2 dt$$

$$|\chi(\tau, \nu)|^2 = \left| \int_{-\infty}^{\infty} s_0(t) s_0^*(t+\tau) e^{j2\pi\nu t} dt \right|^2 \leq \int_{-\infty}^{\infty} |s_0(t)|^2 dt \cdot \int_{-\infty}^{\infty} |s_0^*(t+\tau) e^{j2\pi\nu t}|^2 dt$$

$$\begin{aligned} \int_{-\infty}^{\infty} |s_0(t)|^2 dt \cdot \int_{-\infty}^{\infty} |s_0^*(t+\tau)|^2 dt &= \int_{-\infty}^{\infty} |s_0(t)|^2 dt \cdot \int_{-\infty}^{\infty} |s_0^*(t+\tau) e^{j2\pi\nu t}|^2 dt = \\ &= \int_{-\infty}^{\infty} |s_0(t)|^2 dt \cdot \int_{-\infty}^{\infty} |s_0(t+\tau)|^2 dt = \int_{-\infty}^{\infty} |s_0(t)|^2 dt \cdot \int_{-\infty}^{\infty} |s_0(t)|^2 dt = \chi^2(0,0) = 1 \cdot 1 = 1 \end{aligned}$$

$$|\chi(0,0)| = \left| \int_{-\infty}^{+\infty} s_0(t) s_0^*(t) dt \right| = E \approx 1$$

$$|\chi(\tau, \nu)| \leq |\chi(0,0)| = 1$$

# Funzione di ambiguità (IX)

– **Proprietà #2 della Funzione di Ambiguità:** Volume costante

$$\begin{aligned}
 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi(\tau, \nu)|^2 d\tau d\nu &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} s_0(t) s_0^*(t + \tau) e^{j2\pi\nu t} dt \right|^2 d\tau d\nu = \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} s_0(t_1) s_0^*(t_1 + \tau) e^{j2\pi\nu t_1} dt_1 \right] \left[ \int_{-\infty}^{\infty} s_0^*(t_2) s_0(t_2 + \tau) e^{-j2\pi\nu t_2} dt_2 \right] d\tau d\nu = \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s_0(t_1) s_0^*(t_1 + \tau) e^{j2\pi\nu t_1} s_0^*(t_2) s_0(t_2 + \tau) e^{-j2\pi\nu t_2} dt_1 dt_2 d\tau d\nu = \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s_0(t_1) s_0^*(t_1 + \tau) s_0^*(t_2) s_0(t_2 + \tau) \left[ \int_{-\infty}^{\infty} e^{j2\pi\nu(t_1 - t_2)} d\nu \right] dt_1 dt_2 d\tau = \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s_0(t_1) s_0^*(t_1 + \tau) s_0^*(t_2) s_0(t_2 + \tau) \delta(t_1 - t_2) dt_1 dt_2 d\tau = \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s_0(t_1) s_0^*(t_1 + \tau) s_0^*(t_1) s_0(t_1 + \tau) dt_1 d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |s_0(t_1)|^2 |s_0(t_1 + \tau)|^2 dt_1 d\tau = \\
 &= \int_{-\infty}^{\infty} |s_0(t_1)|^2 \left[ \int_{-\infty}^{\infty} |s_0(t_1 + \tau)|^2 d\tau \right] dt_1 = \int_{-\infty}^{\infty} |s_0(t_1)|^2 dt_1 = 1 \iff \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi(\tau, \nu)|^2 d\tau d\nu = 1
 \end{aligned}$$

Handwritten note in blue ink:

$$\int_{-\infty}^{+\infty} f(t) \delta(t - \tau) dt = f(\tau)$$

Arrows point from the variables  $t_1$  and  $t_2$  in the main derivation to the  $t$  in the note.

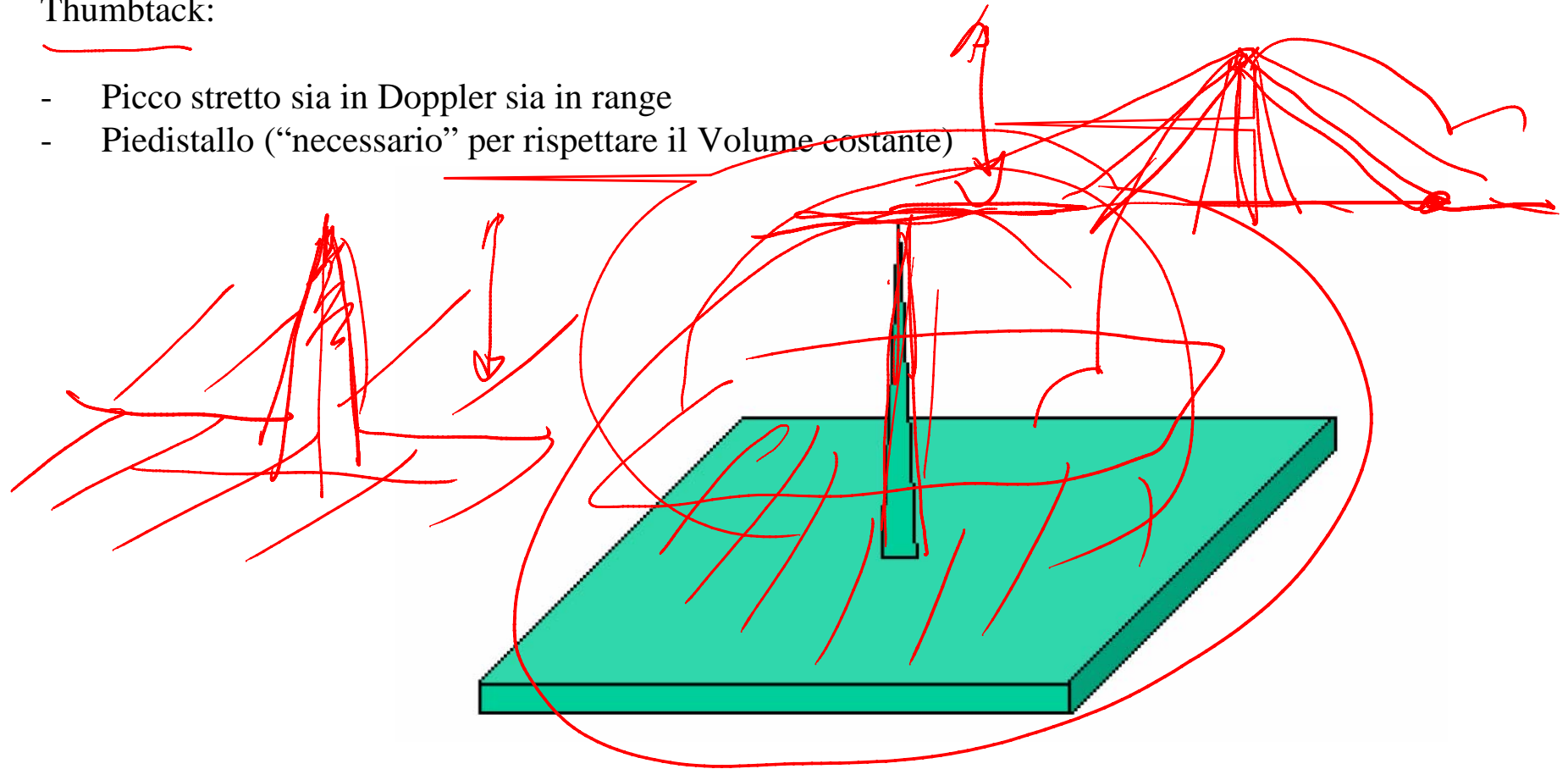
Handwritten red notes:  $E = 1$

# Funzione di ambiguità (X)

## – Funzione di Ambiguità ideale

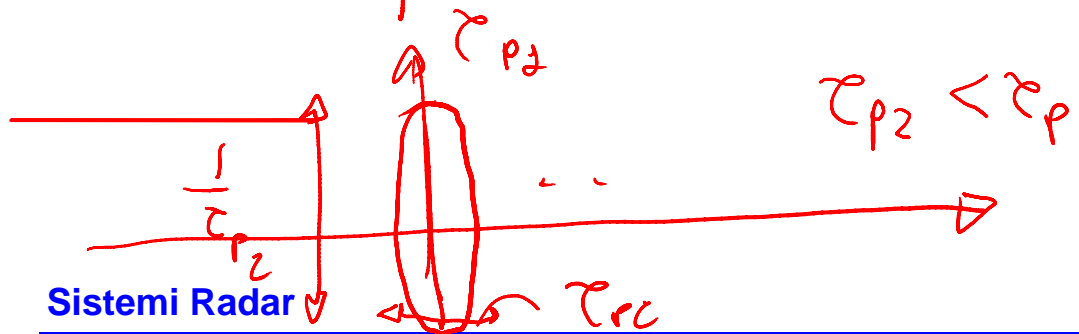
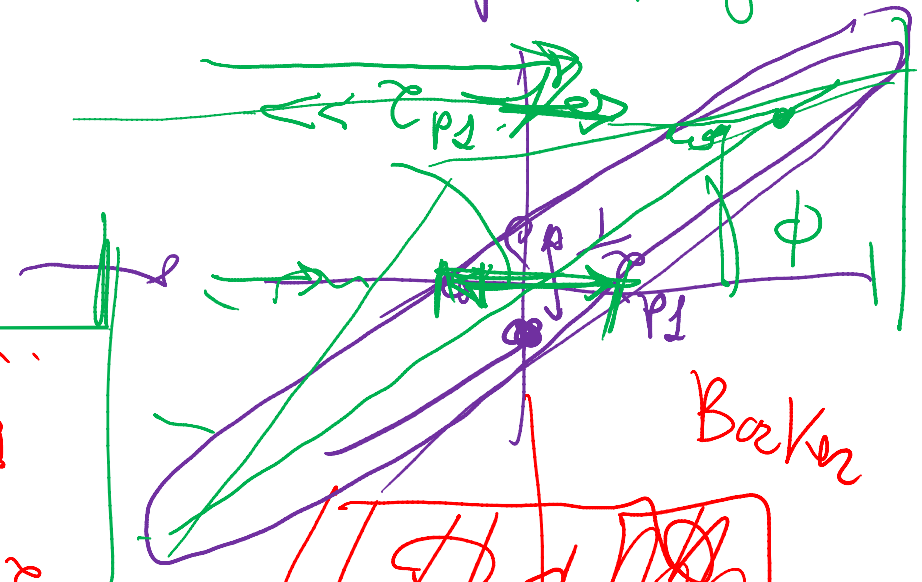
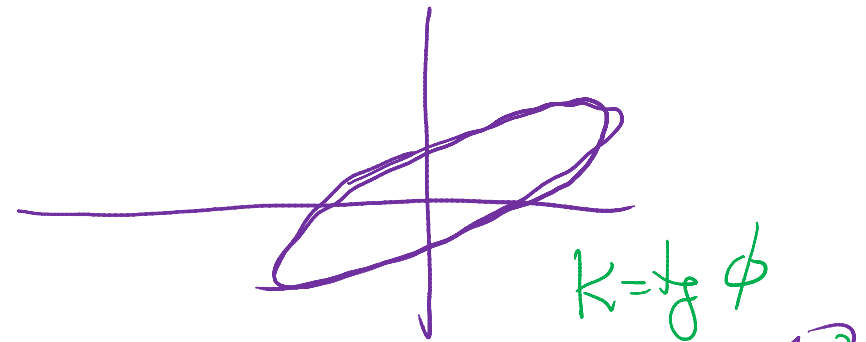
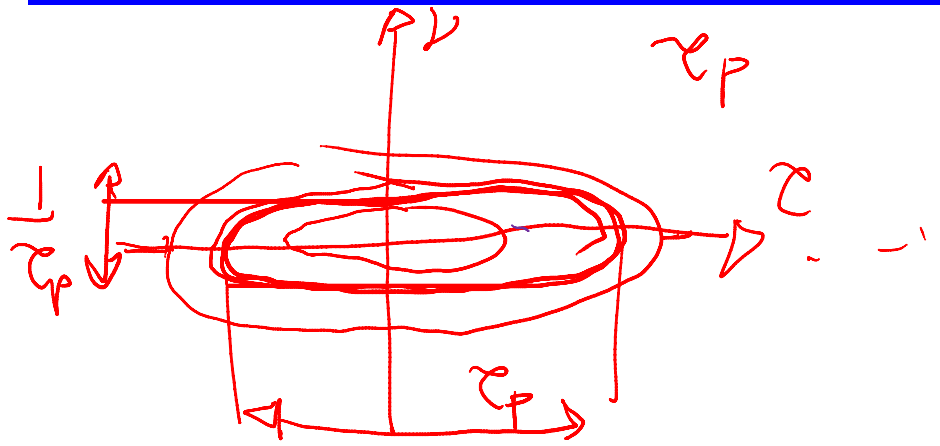
Thumbtack:

- Picco stretto sia in Doppler sia in range
- Piedistallo (“necessario” per rispettare il Volume costante)



rect

rect  $e^{i\pi k^2}$



Sistemi Radar



# Funzione di ambiguità (XI)

- **Proprietà #3 della Funzione di Ambiguità:** Simmetria rispetto all'origine

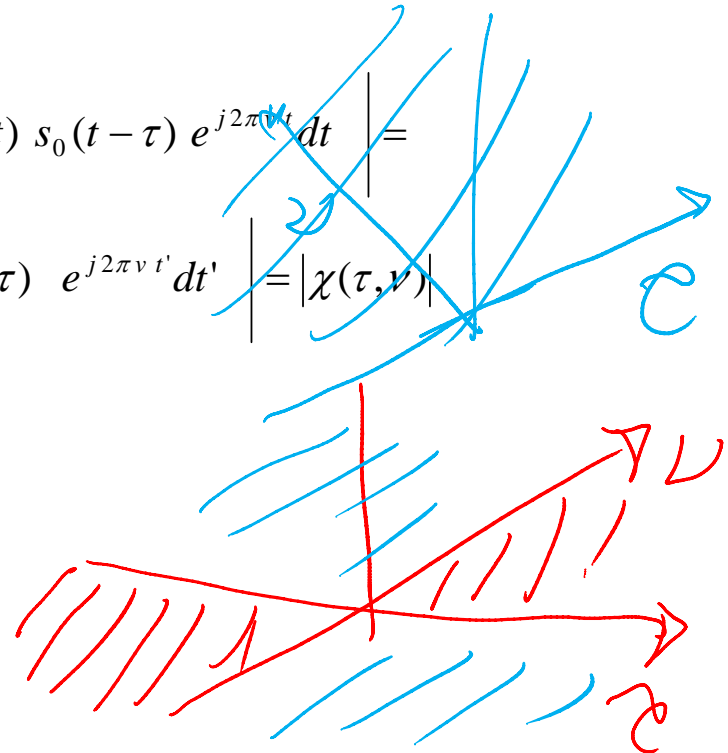
Ricordando che il modulo di una funzione coincide con il modulo della funzione coniugata :

$$\begin{aligned}
 |\chi(-\tau, -\nu)| &= \left| \int_{-\infty}^{\infty} s_0(t) s_0^*(t-\tau) e^{-j2\pi\nu t} dt \right| = \left| \int_{-\infty}^{\infty} s_0^*(t) s_0(t-\tau) e^{j2\pi\nu t} dt \right| = \\
 &= \left| \int_{-\infty}^{\infty} s_0^*(t'+\tau) s_0(t') e^{j2\pi\nu(t'+\tau)} dt' \right| = \left| \int_{-\infty}^{\infty} s_0(t') s_0^*(t'+\tau) e^{j2\pi\nu t'} dt' \right| = |\chi(\tau, \nu)|
 \end{aligned}$$

$$t = t' + \tau$$



$$|\chi(-\tau, -\nu)| = |\chi(\tau, \nu)|$$





# Funzione di ambiguità (XII)

– **Proprietà #4 della Funzione di Ambiguità:** Modulazione lineare di frequenza

$$|\chi(\tau, \nu)| = \left| \int_{-\infty}^{\infty} s_0(t) s_0^*(t + \tau) e^{j2\pi\nu t} dt \right|$$

$s_0(t) \leftrightarrow |\chi(\tau, \nu)|$

$$|\chi_1(\tau, \nu)| = \left| \int_{-\infty}^{\infty} \underbrace{s_0(t) e^{j\pi k t^2}}_{s_1(t)} \underbrace{s_0^*(t + \tau) e^{-j\pi k (t + \tau)^2}}_{s_1^*(t + \tau)} e^{j2\pi\nu t} dt \right| =$$

$s_1(t) = s_0(t) e^{j\pi k t^2}$

$$= \left| \int_{-\infty}^{\infty} s_0(t) s_0^*(t + \tau) e^{-j\pi k \tau^2} e^{-j2\pi k \tau t} e^{j2\pi\nu t} dt \right| =$$

$|\chi_1(\tau, \nu)|^2$

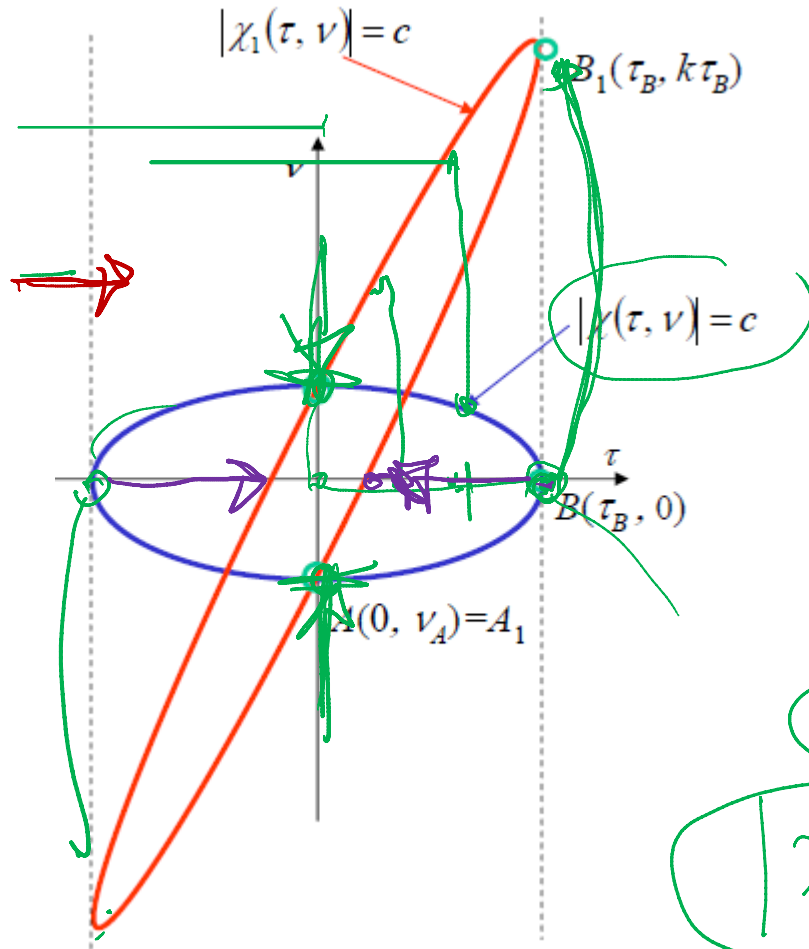
$$= \left| e^{-j\pi k \tau^2} \int_{-\infty}^{\infty} s_0(t) s_0^*(t + \tau) e^{j2\pi(\nu - k\tau)t} dt \right| =$$

$j\pi k [t^2 - (t^2 + \tau^2 + 2\tau t)]$

$$= \left| \int_{-\infty}^{\infty} s_0(t) s_0^*(t + \tau) e^{j2\pi(\nu - k\tau)t} dt \right| = |\chi(\tau, \nu - k\tau)|$$

$$\begin{matrix} s_0(t) & \Leftrightarrow & |\chi(\tau, \nu)| \\ \Leftrightarrow & & \\ s_0(t) e^{j\pi k t^2} & \Leftrightarrow & |\chi(\tau, \nu - k\tau)| \end{matrix}$$

# Funzione di ambiguità (XIII)



$$|\chi_1(\tau_b, k\tau_b)| = |\chi(\tau_b, k\tau_b - k\tau_b)| = |\chi(\tau_b, 0)|$$

$$|\chi_1(\tau, \nu)| = |\chi(\tau, \nu - k\tau)| \neq c$$

$$\nu = 0$$

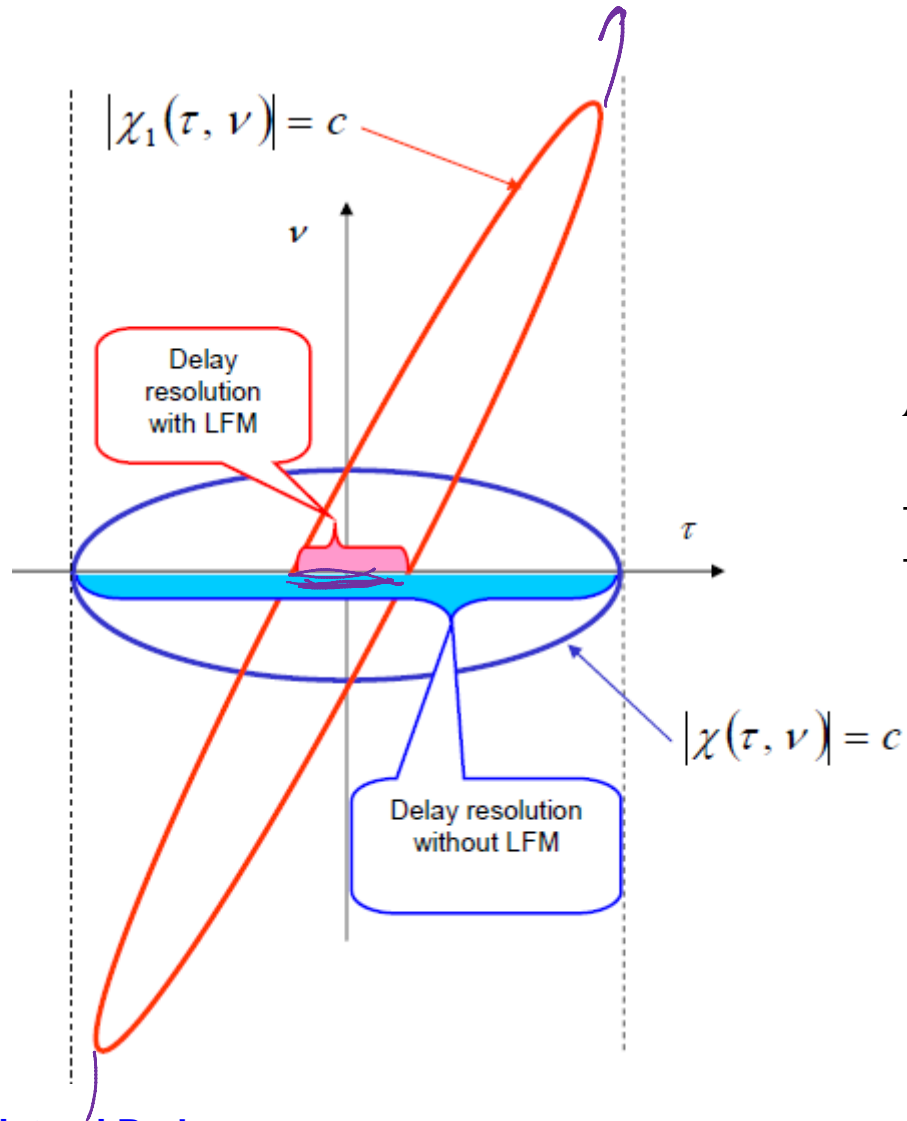
$$|\chi(\tau, 0)| = c$$

$$\nu - k\tau = 0$$

↓

$$\nu = k\tau$$

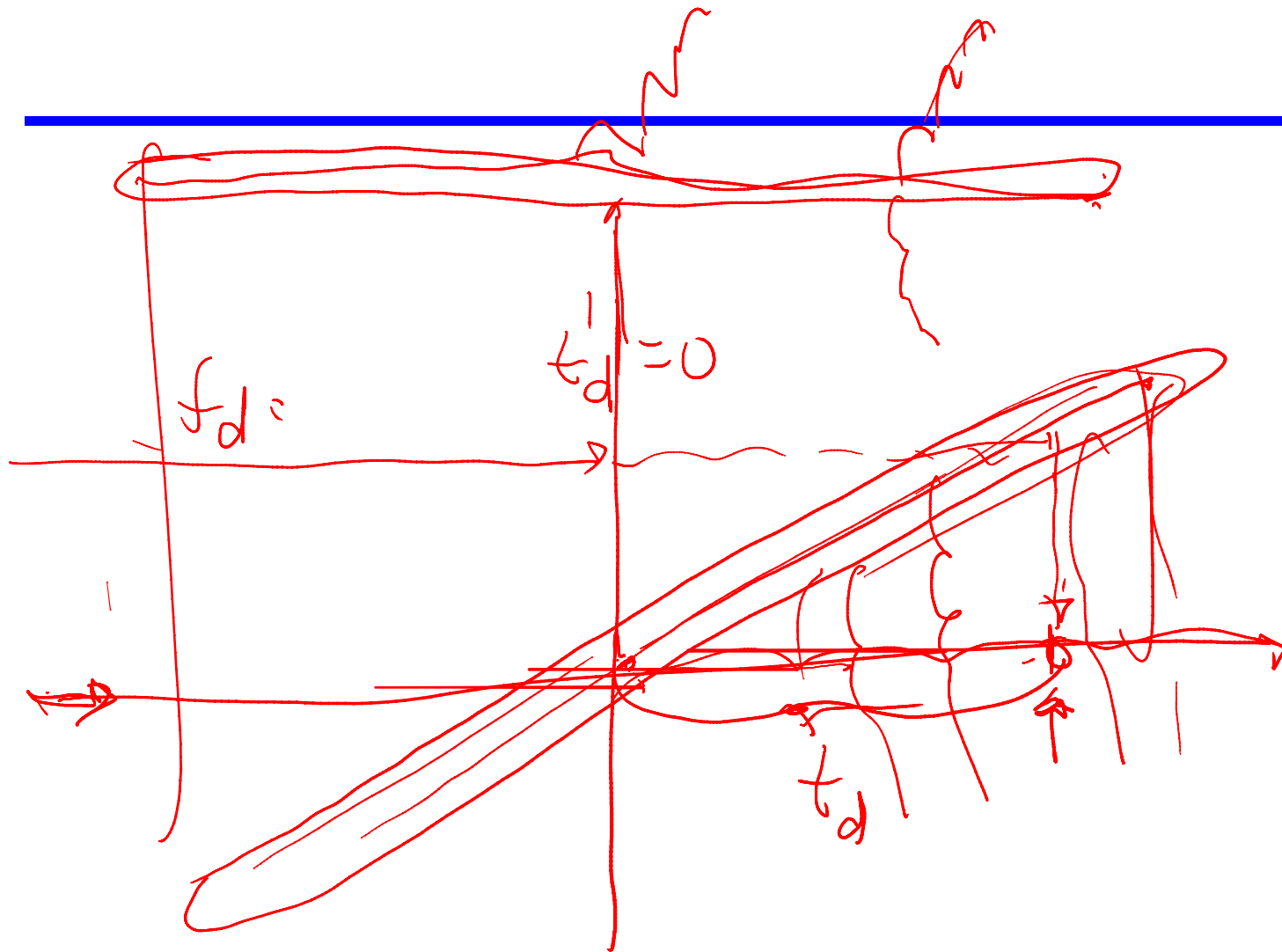
# Funzione di ambiguità (XIV)



$$|\chi_1(\tau, \nu)| = |\chi(\tau, \nu - k\tau)|$$

Aggiunta di modulazione lineare di frequenza:

- Incremento della banda di frequenza
- Riduzione della finestra in distanza



## Sistemi Radar

# Funzione di ambiguità del Chirp (I)

- **Funzione di Ambiguità:**  
Chirp con involucro rettangolare

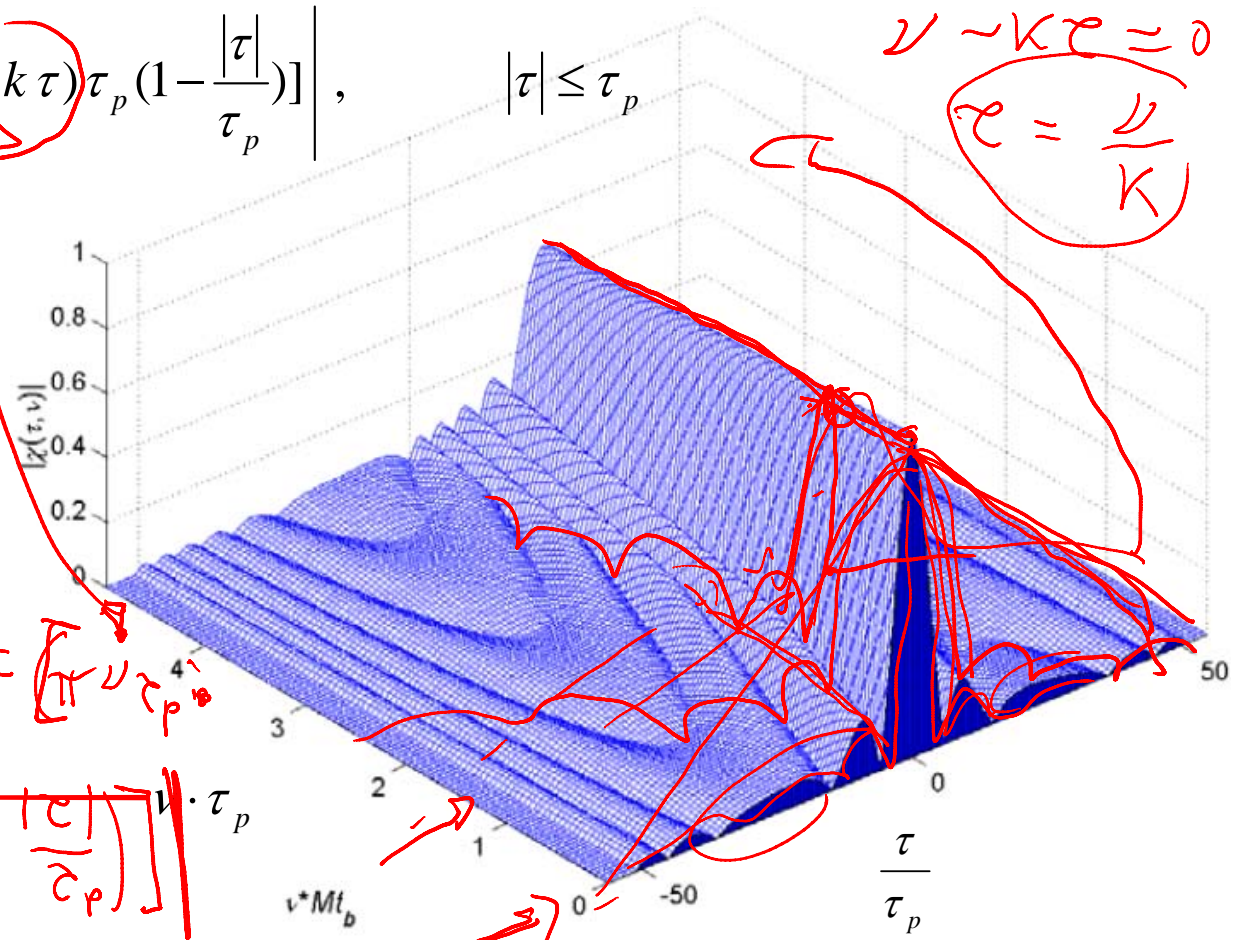
$$s_0(t) = \frac{1}{\sqrt{\tau_p}} \text{rect}_{\tau_p}(t) e^{j\pi k t^2}$$

$$|\chi(\tau, \nu)| = \left| \left(1 - \frac{|\tau|}{\tau_p}\right) \text{sinc}[\pi(\nu - k\tau)\tau_p \left(1 - \frac{|\tau|}{\tau_p}\right)] \right|, \quad |\tau| \leq \tau_p$$

Partendo dalla FA dell'impulso non modulato e usando la proprietà #4:

$$|\chi(\tau, \nu)| = \left(1 - \frac{|\tau|}{\tau_p}\right) \text{sinc}[\pi(\nu - k\tau)\tau_p \left(1 - \frac{|\tau|}{\tau_p}\right)]$$

rect



# Funzione di ambiguità del chirp (II)

Funzione di Ambiguità: Chirp con involuppo rettangolare

$$s_0(t) = \frac{1}{\sqrt{\tau_p}} \text{rect}_{\tau_p}(t) e^{j\pi k t^2}$$

$$|\chi(\tau, 0)| = \left| \left(1 - \frac{|\tau|}{\tau_p}\right) \text{sinc} \left[ \pi k \tau \tau_p \left(1 - \frac{|\tau|}{\tau_p}\right) \right] \right|, \quad |\tau| \leq \tau_p$$

Primo nullo

$$\pi k \tau \tau_p \left(1 - \frac{|\tau|}{\tau_p}\right) = \pi$$

$$\tau \tau_p - \tau^2 = \frac{1}{k}$$

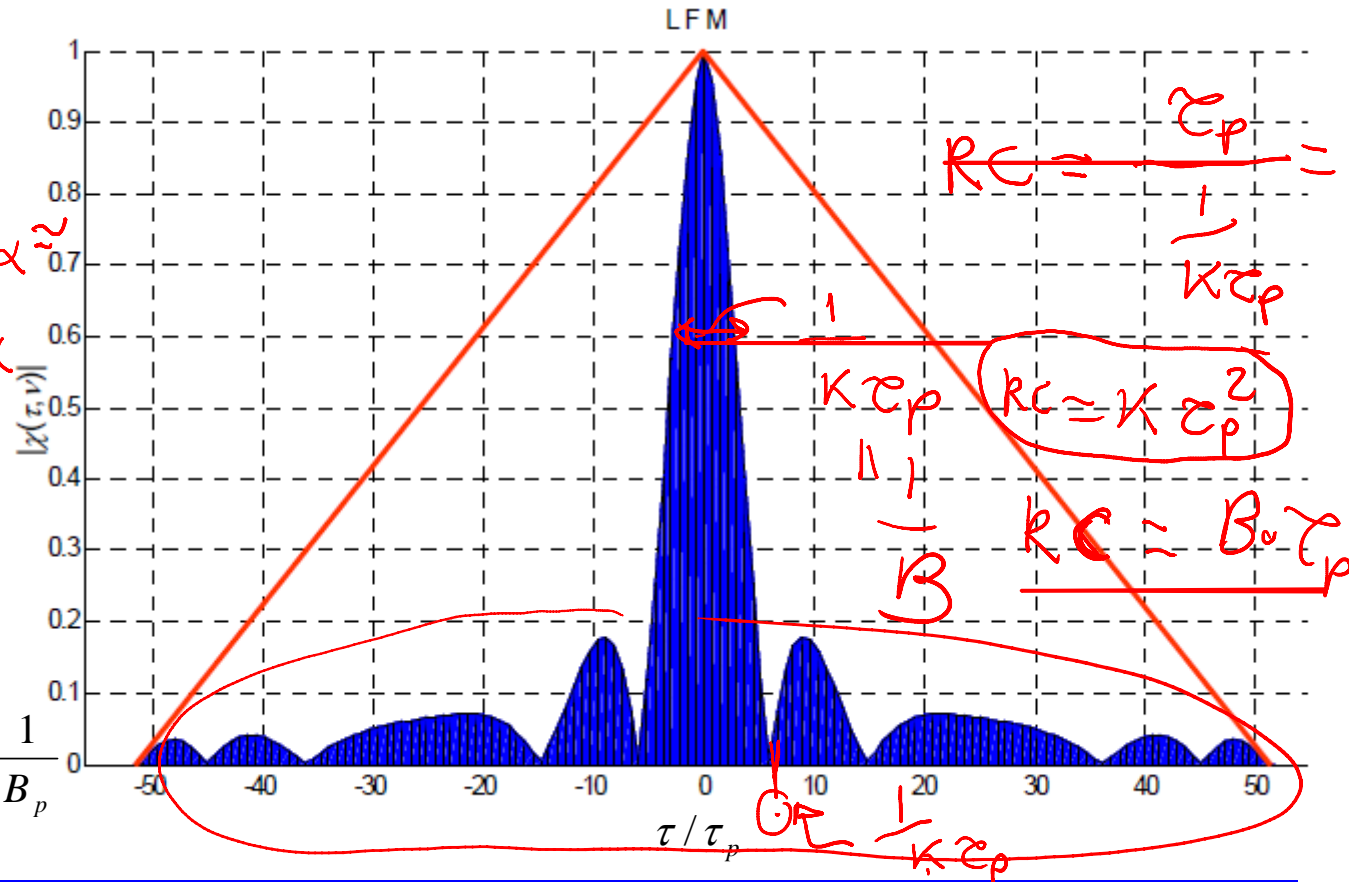
$$\tau^2 - \tau \tau_p + \frac{1}{k} = 0$$

$$\tau = \frac{\tau_p}{2} - \sqrt{\frac{\tau_p^2}{4} - \frac{1}{k}}$$

$$= \frac{\tau_p}{2} - \frac{\tau_p}{2} \sqrt{1 - \frac{4}{k\tau_p^2}}$$

$$\approx \frac{\tau_p}{2} - \frac{\tau_p}{2} \left(1 - \frac{2}{k\tau_p^2}\right) = \frac{1}{k\tau_p} = \frac{1}{B_p}$$

$$\sqrt{1-x} \approx 1 - \frac{x}{2}$$



Sistemi Radar

# Funzione di ambiguità del chirp (III)

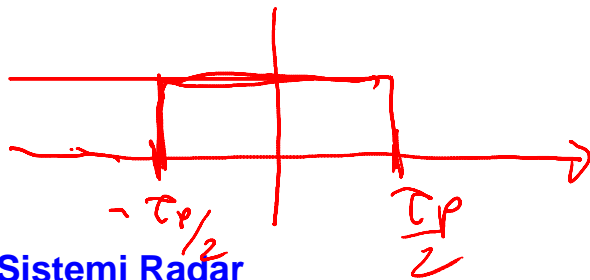
Funzione di Ambiguità: Chirp con involuppo rettangolare

$$s_0(t) = \frac{1}{\sqrt{\tau_p}} \text{rect}_{\tau_p}(t) e^{j\pi k t^2}$$

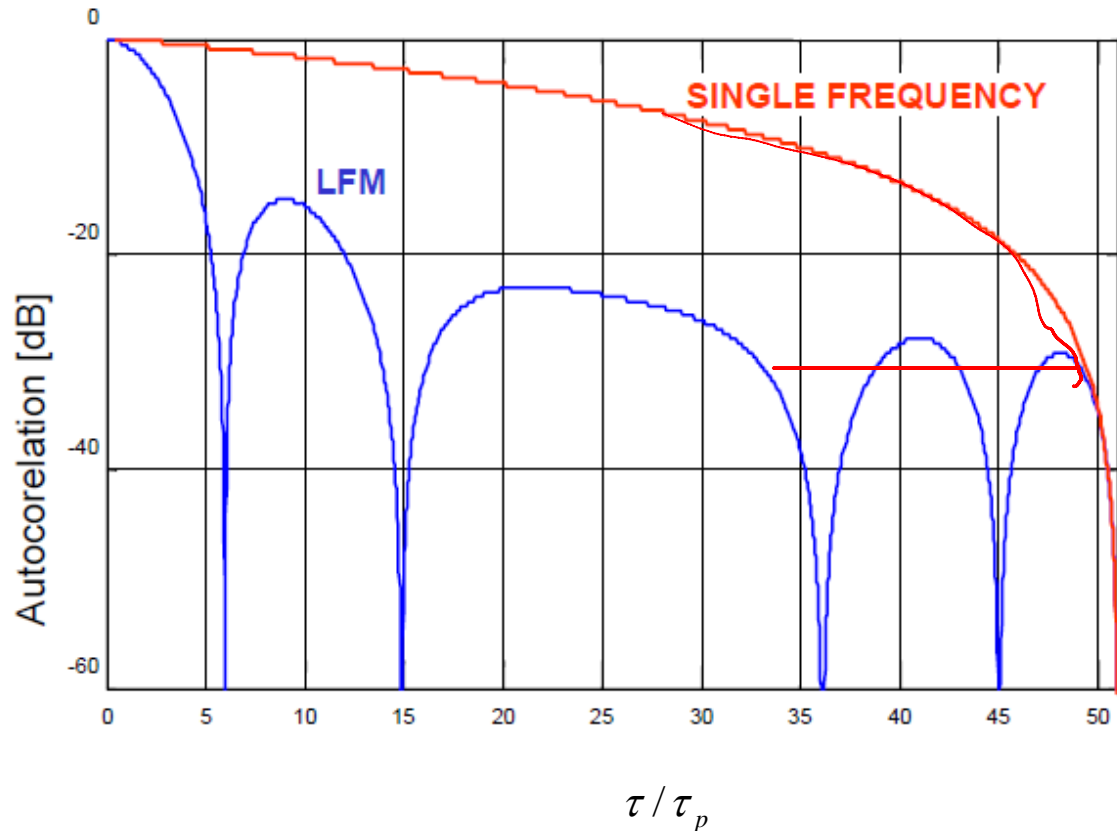
$$|\chi(\tau, 0)| = \left| \left(1 - \frac{|\tau|}{\tau_p}\right) \text{sinc}\left[\pi k \tau \tau_p \left(1 - \frac{|\tau|}{\tau_p}\right)\right] \right|, \quad |\tau| \leq \tau_p$$

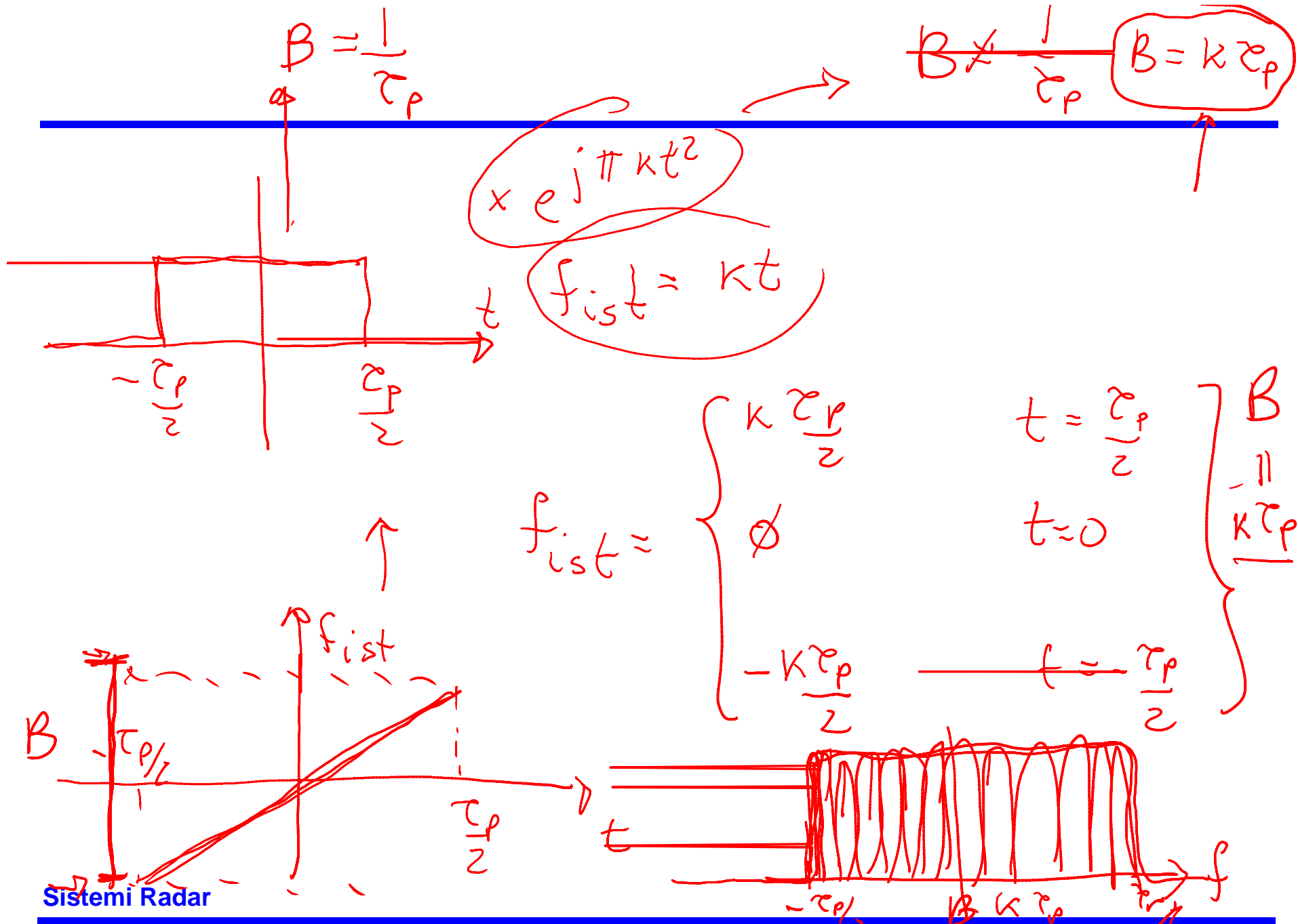
Rapporto di compressione

$$\frac{\tau_p}{1} = k \tau_p^2 = B_p \tau_p$$



Sistemi Radar







# Funzione di ambiguità del chirp (IV)

Funzione di Ambiguità: Chirp con involuppo rettangolare

$$s_0(t) = \frac{1}{\sqrt{\tau_p}} \text{rect}_{\tau_p}(t) e^{j\pi k t^2}$$

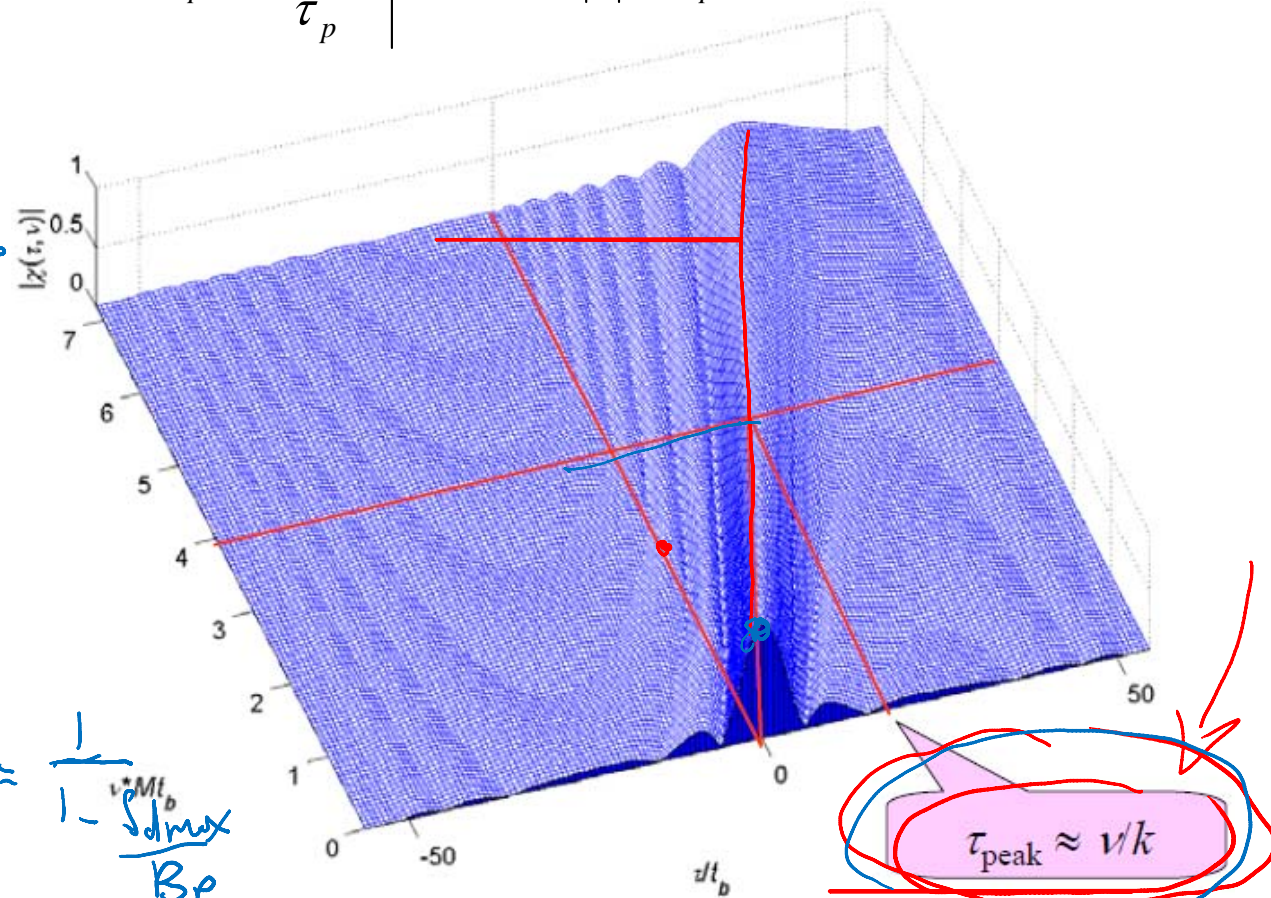
$$|\chi(\tau, 0)| = \left| \underbrace{\left(1 - \frac{|\tau|}{\tau_p}\right)}_{\text{handwritten}} \sin c\left[\pi k \tau \tau_p \left(1 - \frac{|\tau|}{\tau_p}\right)\right]\right|, \quad |\tau| \leq \tau_p$$

$$\left(1 - \frac{|\tau|}{\tau_p}\right) \approx 1 - \frac{v}{k\tau_p}$$

1                      1  
 Accoppiamento  
 fra ritardo e  
 Doppler usando  
 LFM

$$L_{\text{oss}} = \frac{1}{1 - \frac{v}{k\tau_p}} = \frac{1}{1 - \frac{v^* M t_b}{B_p}}$$

Sistemi Radar



# Esercizio

$$\tau_p = 50 \mu s$$

blind range  $\frac{c \tau_p}{2} = \frac{3 \cdot 10^8 \cdot 50 \cdot 10^{-6}}{2} = \frac{15000}{2} = 7500 \text{ m}$

$$r_{is} = 150 \text{ m}$$

$$R_c = \frac{7500}{150} = 50 = B_p \cdot \tau_p = k \tau_p^2 \quad \left| \quad r_{is} = \frac{c}{2B}$$

$$B_p = 1 \text{ MHz} \approx \frac{50}{50 \mu s}$$

$$B_p = k \tau_p$$



$$k \approx \frac{B_p}{\tau_p}$$

$$B = \frac{c}{2 r_{is}} = \frac{3 \cdot 10^8}{2 \cdot 150}$$

$$k = \frac{10^6 \text{ Hz}}{50 \cdot 10^{-6} \text{ s}} = 2 \cdot 10^{10} \text{ s}^{-2}$$

$$e^{i \pi k t^2}$$

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$$\text{Error in temp} = \frac{V}{R}$$

— allora  $V_{\max} = 300 \text{ m/s}$

$$f = 1,5 \text{ GHz} \rightarrow \lambda = \frac{c}{f} = 0,2 \text{ m}$$

$$f_{d\max} = \frac{2V_{\max}}{\lambda} = \frac{2 \cdot 300 \text{ m/s}}{0,2 \text{ m}} = 2 \cdot 1500 \text{ Hz} = 3000 \text{ Hz}$$

$$\text{error max in temp} = \frac{3000 \text{ Hz}}{2 \cdot 10^{10} \text{ s}^{-2}} = 1,5 \cdot 10^{-7} \text{ s}$$

$$\text{error max in range} = \frac{3 \cdot 10^8}{2} \cdot 1,5 \cdot 10^{-7} \text{ m/s} = 22,5 \text{ m}$$

$$\frac{1}{1 - \left(\frac{f_{dmax}}{B_p}\right)^2}$$

RR

$$\frac{1 - \frac{f_{dmax}}{B_p} + \frac{f_{dmax}}{B_p}}{1 - \frac{f_{dmax}}{B_p}} \approx 1 + \frac{\frac{f_{dmax}}{B_p}}{1 - \frac{f_{dmax}}{B_p}} \approx \left(1 + \frac{f_{dmax}}{B_p} \approx 1 + 3 \cdot 10^{-3}\right)$$

trascurabile!

$$f_{dmax} = 3 \text{ kHz}$$

$$B_p = 2 \text{ MHz}$$

$$\frac{f_{dmax}}{B_p} = \frac{3 \cdot 10^3}{2 \cdot 10^6} \approx 3 \cdot 10^{-3}$$