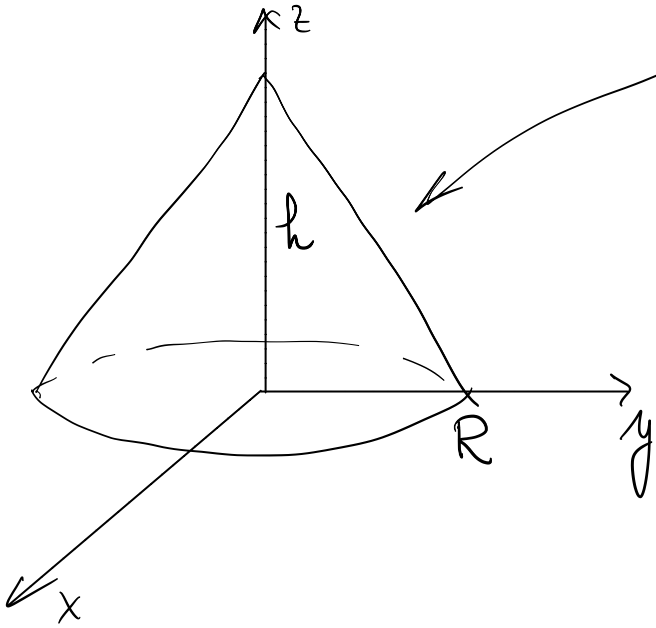
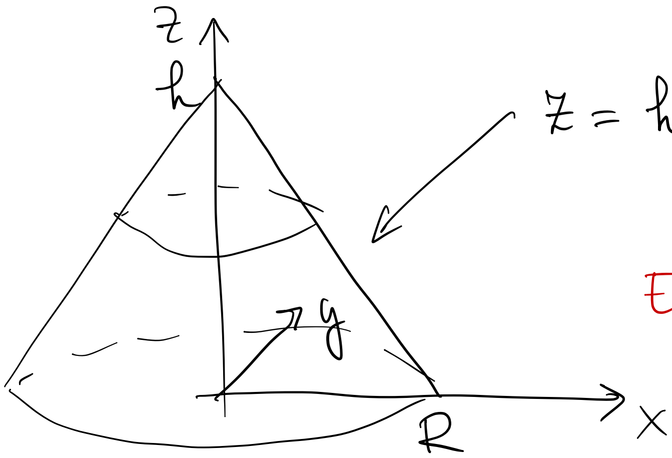
Esempio: Baricentro del cono circolare retto.



equazione della "falda" che ricopre il cono.



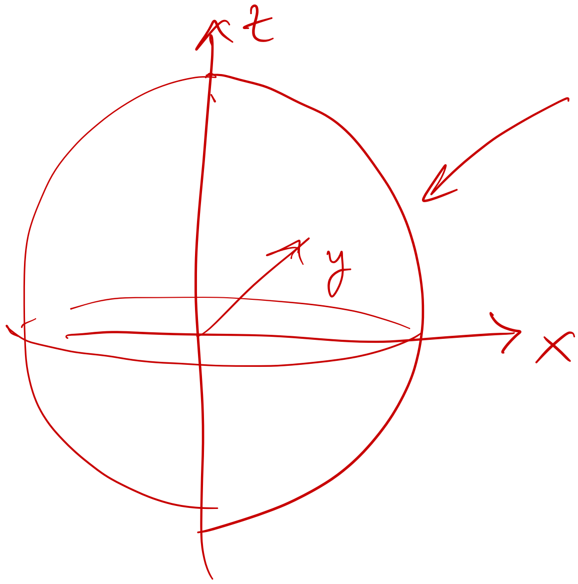
$$z = h - \frac{h}{R}x = \frac{h}{R}(R-x)$$

$$x \longrightarrow \sqrt{x^2+y^2}$$

Eq^{ne} del cono

$$z = \frac{h}{R}\left(R-\sqrt{x^2+y^2}\right)$$

$\Longrightarrow$ in coord. cilindriche $z = \frac{h}{R}(R-\rho)$



$z^2+x^2=R^2$ (opp $x=\sqrt{R^2-z^2}$)

$z^2+x^2+y^2=R^2$ equaz sfera.

Il cono "pieno" descritto sopra si scrive, in coord. cartesiane

$$E = \left\{ (x,y,z) : \ x^2+y^2 \le R^2, \qquad 0 \le z \le \frac{h}{R}\left(R - \sqrt{x^2+y^2}\right) \right\}$$

in coord. cilindriche diventa

$$\tilde{E} = \left\{ (\rho, \theta, z) : 0 \le \theta \le 2\pi, \quad 0 \le \rho \le R, \quad 0 \le z \le \frac{h}{R}(R-\rho) \right\}$$

$$z_B = \frac{1}{\mathrm{Vol}\, E} \iiint_E z \, dx\, dy\, dz$$

$$\mathrm{Vol}\, E = \iiint_E 1 \, dx\, dy\, dz = \int_0^{2\pi} d\theta \int_0^R d\rho \int_0^{\frac{h}{R}(R-\rho)} dz \ \rho =$$

$$= 2\pi \int_0^R d\rho \ \rho \ \frac{h}{R}(R-\rho) = \frac{2\pi h}{R}\left(R \cdot \frac{R^2}{2} - \frac{R^3}{3} \right) =$$

$$= \frac{2\pi h R^2}{6} = \frac{\pi R^2 h}{3}$$

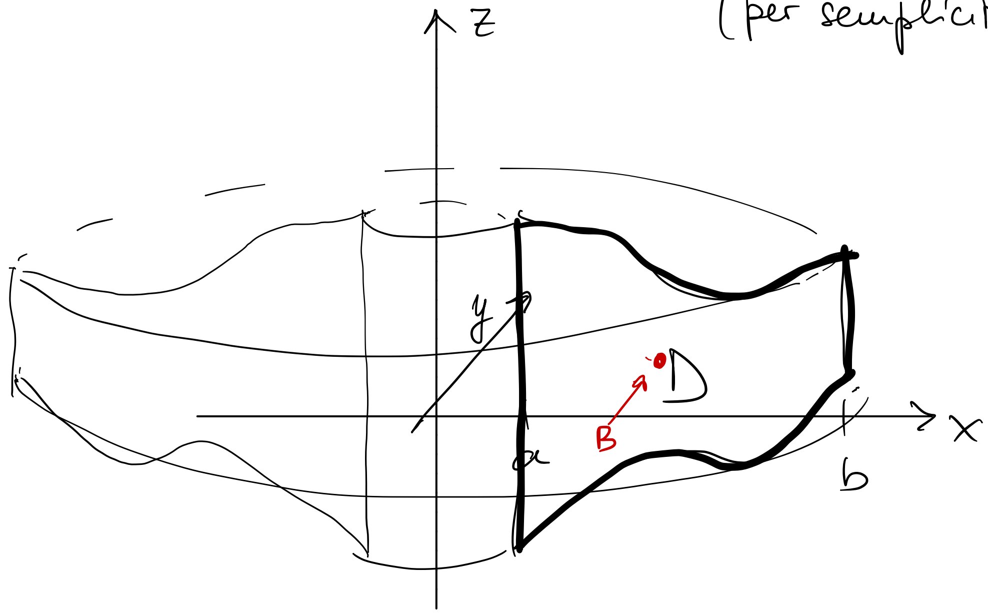
$$\iiint_E z \, dx\, dy\, dz = \int_0^{2\pi} d\theta \int_0^R d\rho \int_0^{\frac{h}{R}(R-\rho)} dz \ z \rho =$$

$$= \frac{2\pi}{2} \int_0^R d\rho \ \rho \ \frac{h^2}{R^2} (R-\rho)^2 = \pi \frac{h^2}{R^2} \int_0^R d\rho \ (R-\rho)^2 \rho =$$

$$= \frac{\pi h^2}{R^2} \int_0^R \left(R^2 \rho + \rho^3 - 2R\rho^2 \right) d\rho = \ldots$$

Volume dei solidi di rotazione

Sia D un dominio normale del semipiano xz, $x \geq 0$.
(per semplicità, normale risp. alla y)



$$D = \{(x, z) : a \leq x \leq b \,,\, \alpha(x) \leq z \leq \beta(x)\}$$

Sia E il solido di rotazione ottenuto facendo ruotare D di un giro completo intorno all'asse z.

Vol E = ?

In coordinate cilindriche
$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\theta \\ z = z \end{cases}$$

E diventa il dominio

$$\tilde{E} = \{(\rho, \theta, z) : 0 \leq \theta \leq 2\pi,\ a \leq \rho \leq b,\ \alpha(\rho) \leq z \leq \beta(\rho)\}$$

Vol $E = \iint$

$$\mathrm{vol}\,E = \iiint_E 1\,dx\,dy\,dz = \iiint_{\tilde{E}} \rho\,d\theta\,d\rho\,dz =$$

$$= \int_0^{2\pi} d\theta \int_a^b d\rho\,\rho \int_{\alpha(\rho)}^{\beta(\rho)} dz = \left(2\pi \int_a^b d\rho\,\rho\,(\beta(\rho)-\alpha(\rho)) = \right.$$

cambio nome alla variabile $\rho \to x$

$$\left. = 2\pi \int_a^b dx\,x\,(\beta(x)-\alpha(x)) = \right)$$

$$= 2\pi \int_a^b dx \int_{\alpha(x)}^{\beta(x)} dz\,x = 2\pi \iint_D x\,dx\,dz.$$

Questa formula si dimostra anche se D è una qualsiasi unione finita di domini normali

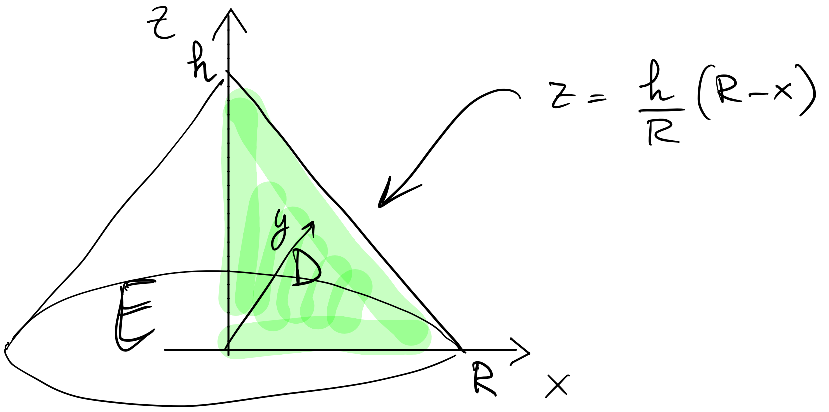
TEOREMA di Guldino per il volume dei solidi di rotazione

Sia D un dominio del semipiano xz, $x \geq 0$. Sia E l'insieme ottenuto facendo ruotare D di un angolo giro intorno all'asse z. Allora

$$\mathrm{vol}\,E = 2\pi \iint_D x\,dx\,dz =$$

$$= \mathrm{Area}(D)\,\underbrace{2\pi \left(\underbrace{\frac{1}{\mathrm{area}(D)} \iint_D x\,dx\,dz}_{x_B = \text{ascissa del baricentro di } D} \right)}_{\text{lunghezza della circonferenza percorsa dal baricentro di D nella sua rotazione.}}$$

Volume del cono (di nuovo)

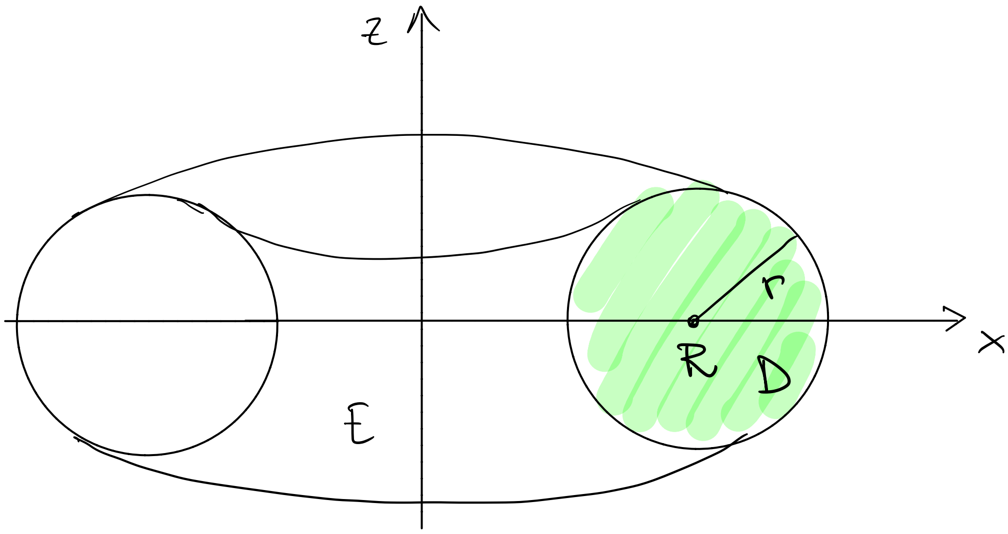


$$\mathrm{Vol}\, E = 2\pi \iint_D x \, dx\, dz = 2\pi \int_0^R dx \, x \int_0^{\frac{h}{R}(R-x)} dz =$$

$$= 2\pi \int_0^R dx \, x \, \frac{h}{R}(R-x) = \frac{2\pi h}{R} \int_0^R dx \left(Rx - x^2\right) =$$

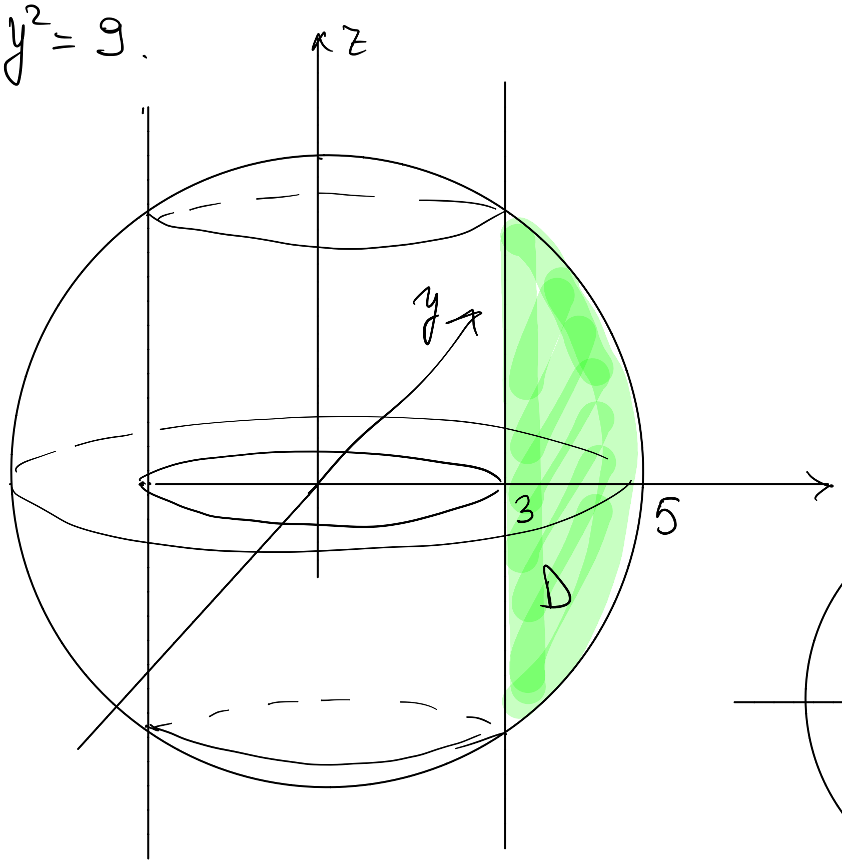
$$= \frac{2\pi h}{R}\left[\frac{R^3}{2} - \frac{R^3}{3}\right] = \frac{2\pi h R^2}{6} = \frac{(\pi R^2)h}{3}$$

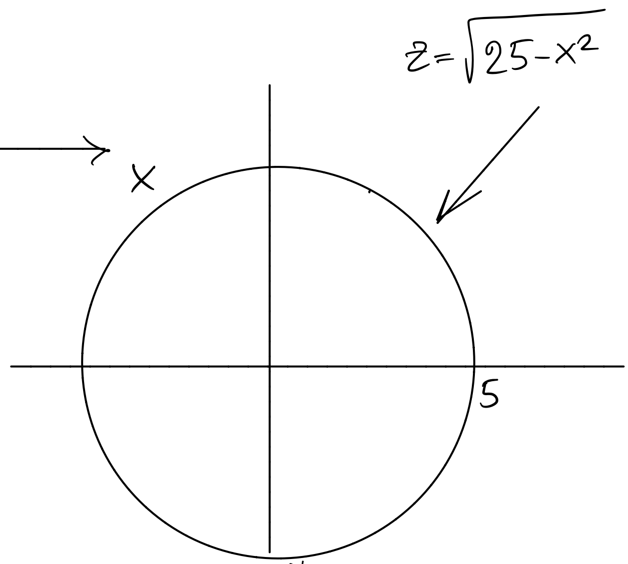
Volume del toro (ciambella)



$$\mathrm{Vol}\, E = (\pi r^2)\, 2\pi R = 2\pi^2 r^2 R$$

Esercizio: Trovare il volume del solido E costituito dai punti interni alla sfera $x^2+y^2+z^2=25$ ma esterni al cilindro $x^2+y^2=9$.





OSS E è di rotazione, ottenuto facendo ruotare il dominio D indicato in verde.

$$\mathrm{Vol}\,E = 2\pi \iint_D x \, dx\, dz = 4\pi \int_3^5 dx\, x \int_0^{\sqrt{25-x^2}} dz =$$

$$= \frac{4\pi}{2} \int_3^5 dx\, 2x \sqrt{25-x^2} = 2\pi \, \frac{2}{3} \left(25-x^2\right)^{3/2} \Big|_{x=5}^{x=3} =$$

$$= \frac{4\pi}{3} \cdot 64 = \frac{256}{3}\pi .$$

Se dovessimo calcolare un integrale triplo su E.
(di una generica $f(x,y,z)$ continua).

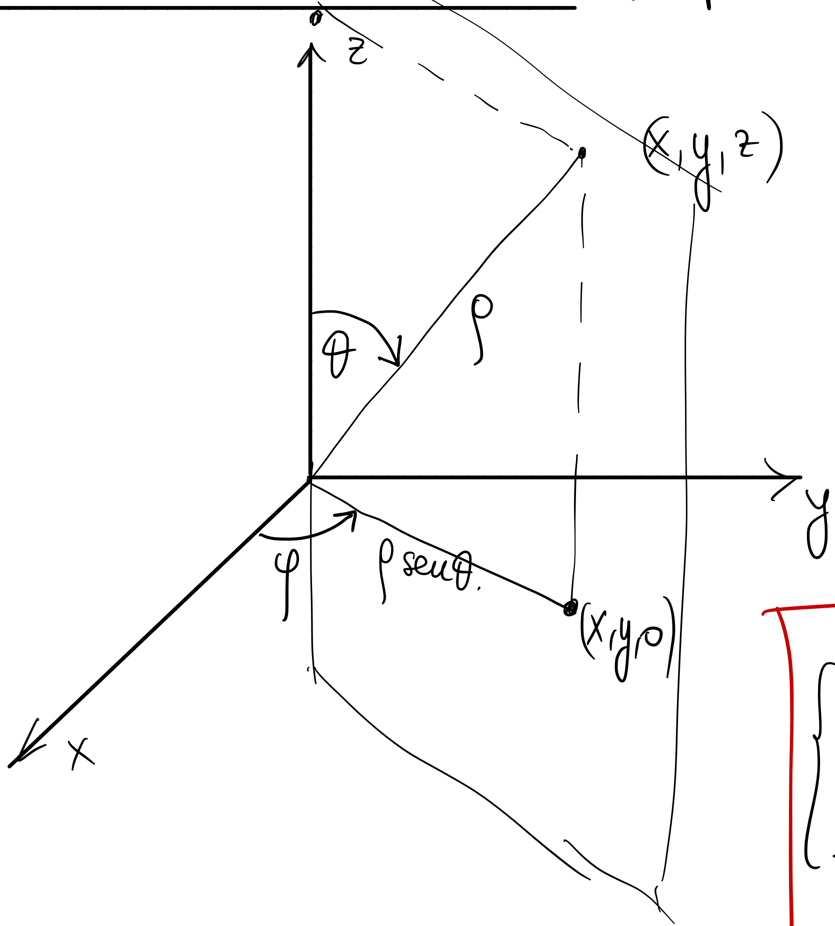
$$\iiint_E f(x,y,z)\,dx\,dy\,dz =$$

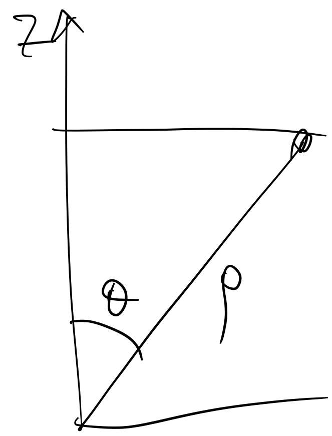
coord. cilindriche

$$= \int_0^{2\pi} d\theta \int_3^5 d\rho \int_{-\sqrt{25-\rho^2}}^{\sqrt{25-\rho^2}} dz \; f(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta, z)\,\rho$$

$$x^2+y^2+z^2=25 \iff \rho^2+z^2=25$$

COORDINATE SFERICHE (o polari in 3D)





$$\boxed{\begin{cases} x = \rho \operatorname{sen}\theta \cos\varphi \\ y = \rho \operatorname{sen}\theta \operatorname{sen}\varphi \\ z = \rho \cos\theta \end{cases}}$$

$$z = \rho \cos\theta$$

$0 \le \rho \le \infty$

$0 \le \theta \le \pi$ colatitudine

$0 \le \varphi \le 2\pi$ longitudine

Anche in questo caso ci sono problemi di iniettività, ma riguardano insiemi di misura nulla.

$$\begin{cases} x = \rho\, \text{sen}\,\theta \cos\varphi \\ y = \rho\, \text{sen}\,\theta\, \text{sen}\,\varphi \\ z = \rho \cos\theta \end{cases}$$

$$\det \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \det \begin{pmatrix} \text{sen}\,\theta\cos\varphi & \rho\cos\theta\cos\varphi & -\rho\,\text{sen}\,\theta\,\text{sen}\,\varphi \\ \text{sen}\,\theta\,\text{sen}\,\varphi & \rho\cos\theta\,\text{sen}\,\varphi & \rho\,\text{sen}\,\theta\cos\varphi \\ \cos\theta & -\rho\,\text{sen}\,\theta & 0 \end{pmatrix} =$$

$$= \rho^2\,\text{sen}\,\theta \quad \det \begin{pmatrix} \text{sen}\,\theta\cos\varphi & \cos\theta\cos\varphi & -\text{sen}\,\varphi \\ \text{sen}\,\theta\,\text{sen}\,\varphi & \cos\theta\,\text{sen}\,\varphi & \cos\varphi \\ \cos\theta & -\text{sen}\,\theta & 0 \end{pmatrix}$$

$$= \rho^2\,\text{sen}\,\theta\left(\cos^2\theta\cos^2\varphi + \text{sen}^2\theta\,\text{sen}^2\varphi + \cos^2\theta\,\text{sen}^2\varphi + \text{sen}^2\theta\cos^2\varphi\right).$$

$$= \rho^2\,\text{sen}\,\theta\left(\cos^2\theta + \text{sen}^2\theta\right) = \rho^2\,\text{sen}\,\theta \geq 0$$

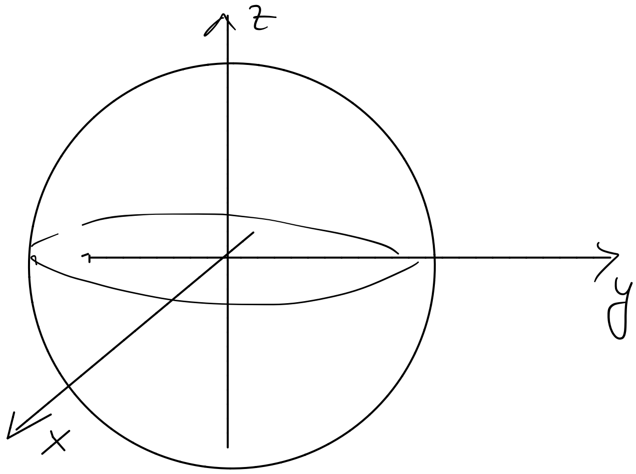
Formula del passaggio a coord. sferiche

$$\iiint_D f(x,y,z)\,dx\,dy\,dz =$$

$$= \iiint_T f(\rho\,\text{sen}\,\theta\cos\varphi,\ \rho\,\text{sen}\,\theta\,\text{sen}\,\varphi,\ \rho\cos\theta)\,\rho^2\,\text{sen}\,\theta\ d\rho\,d\theta\,d\varphi.$$

Volume di una palla in coord sferiche.

$B_R = \{(x,y,z) : \quad x^2+y^2+z^2 \leq R^2\}$



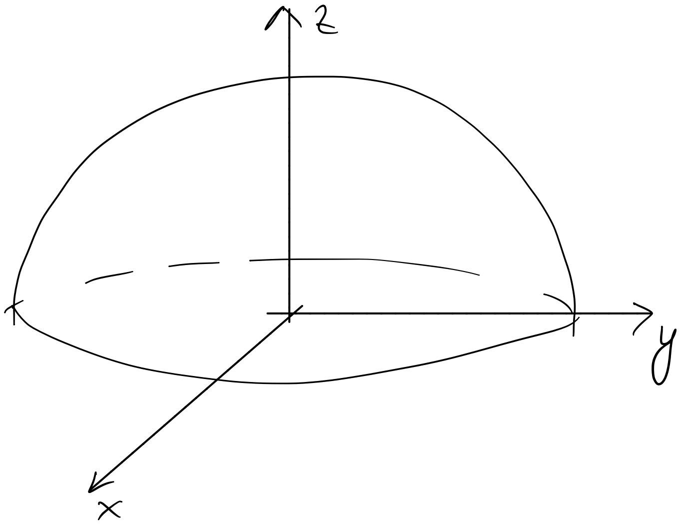
In coord. sferiche B_R diventa

$$\tilde{B} = \{(\rho,\theta,\varphi) : \quad \begin{array}{l} 0 \leq \rho \leq R \\ 0 \leq \theta \leq \pi \\ 0 \leq \varphi \leq 2\pi \end{array}\}$$

$$\text{vol } B_R = \iiint_{B_R} 1 \, dx\,dy\,dz = \int_0^R d\rho \int_0^\pi d\theta \underbrace{\boxed{\int_0^{2\pi} d\varphi}}_{2\pi} \rho^2 \operatorname{sen}\theta =$$

$$= \frac{R^3}{3} 2\pi \cdot 2 = \frac{4\pi}{3} R^3$$

Baricentro di una semipalla in coord. sferiche.



$$B_R^+ = B_R \cap \{z \geq 0\}$$

$x_B = y_B = 0$ (ricontrollarlo)

$$z_B = \frac{1}{vol(B_R^+)} \iiint_{B_R^+} z \, dx\, dy\, dz =$$

$$= \frac{3}{2\pi R^3} \int_0^{\pi/2} d\theta \underbrace{\int_0^{2\pi} d\varphi}_{= 2\pi} \int_0^R d\rho \; \rho \cos\theta \; \rho^2 \operatorname{sen}\theta =$$

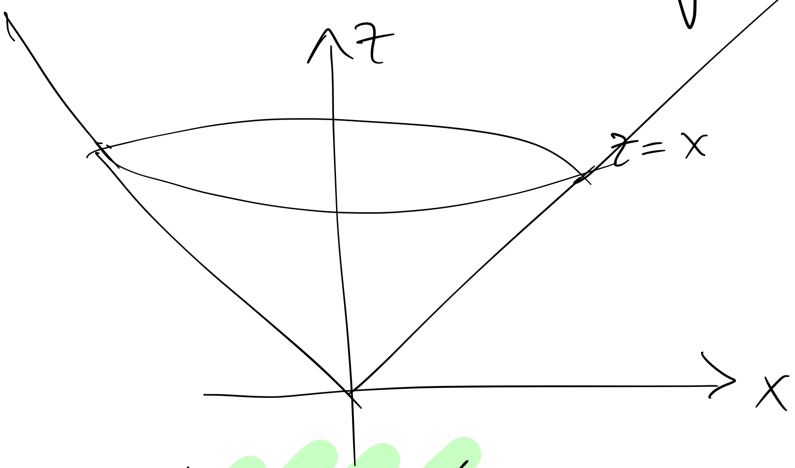
$$= \frac{3}{2\pi R^3} \, 2\pi \left(\int_0^{\pi/2} d\theta \, \cos\theta \operatorname{sen}\theta \right) \cdot \int_0^R d\rho \, \rho^3 =$$

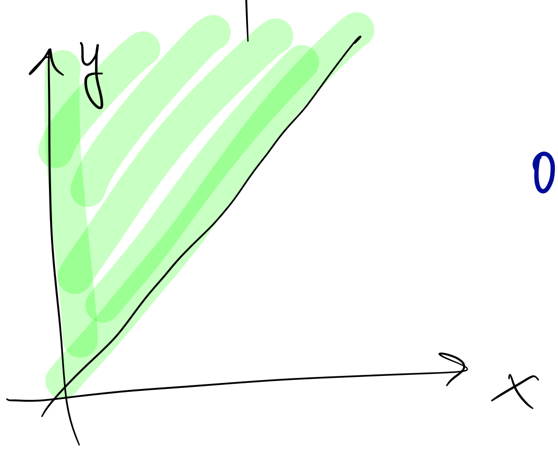
$$= \frac{3}{R^3} \quad \frac{1}{2} \quad \frac{R^4}{4} = \frac{3R}{8}$$

Esercizio. $\displaystyle\iiint_T x^2\,dx\,dy\,dz$

$$T = \{ (x,y,z) : \quad 1 \leq x^2+y^2+z^2 \leq 4, \quad z \geq \sqrt{x^2+y^2}, \quad 0 \leq x \leq y, \}.$$

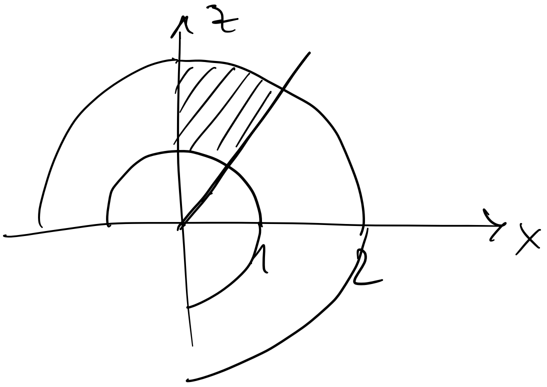
corona sferica





$0 \leq x \leq y$

In coord. polari diventa .... ?



Disegno: la prossima lezione.

In coordinate sferiche.

$$\begin{cases} x = \rho \operatorname{sen}\theta \cos\varphi \\ y = \rho \operatorname{sen}\theta \operatorname{sen}\varphi \\ z = \rho \cos\theta \end{cases}$$

T diventa:

$$\tilde{T} = \left\{ (\rho, \theta, \varphi) : \quad 1 \le \rho \le 2, \quad 0 \le \theta \le \frac{\pi}{4}, \quad \frac{\pi}{4} \le \varphi \le \frac{\pi}{2} \right\}$$

$$\iiint_T x^2 \, dx\,dy\,dz = \int_0^{\pi/4} d\theta \int_1^2 d\rho \int_{\pi/4}^{\pi/2} d\varphi \quad \rho^2 \operatorname{sen}^2\theta \cos^2\varphi \; \rho^2 \operatorname{sen}\theta =$$

$$= \int_0^{\pi/4} d\theta \; \underset{\substack{\parallel \\ (1-\cos^2\theta)\operatorname{sen}\theta}}{\operatorname{sen}^3\theta} \cdot \int_1^2 d\rho \; \rho^4 \cdot \int_{\pi/4}^{\pi/2} d\varphi \; \cos^2\varphi = \underline{\text{quasi immediato}}$$

$t = \cos\theta$