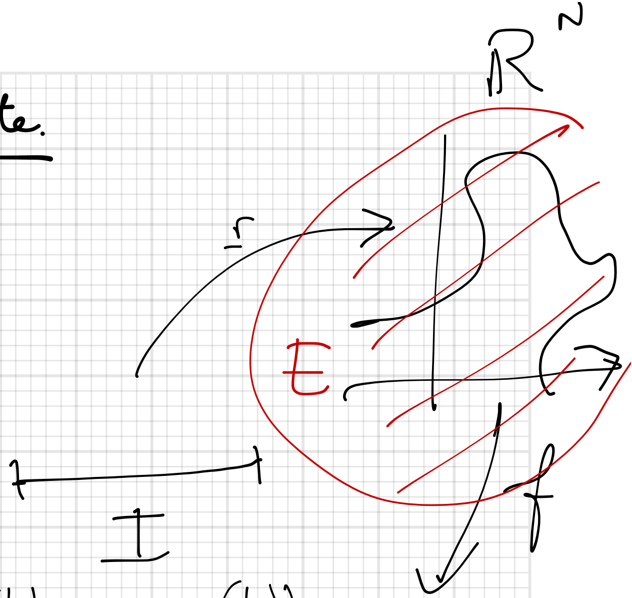
Derivazione di funzioni composte.

$I \subset \mathbb{R}$ intervallo, $E \subset \mathbb{R}^N$ aperto.

$\underline{r} : I \to E$

$t \mapsto \underline{r}(t) = (r_1(t), \dots, r_N(t))$



$f : E \subset \mathbb{R}^N \to \mathbb{R}$, $f(x_1, x_2, \dots x_N)$

$f \circ \underline{r} : I \to \mathbb{R}$

$t \mapsto f(\underline{r}(t)) = f(r_1(t), \dots r_N(t))$

Se $\underline{r}$ è ~~differenziabile~~ derivabile in I

e f è differenziabile in E,

allora $f \circ \underline{r}$ è derivabile in I, e

$$(f \circ r)'(t) = \sum_{i=1}^{N} f_{x_i}(\underline{r}(t))\, r_i'(t) = \nabla f(\underline{r}(t)) \cdot \underline{r}'(t)$$

$$= f_{x_1}(\underline{r}(t))\, r_1'(t) + f_{x_2}(\underline{r}(t))\, r_2'(t) + \dots + f_{x_N}(\underline{r}(t))\, r_N'(t)$$

CASO GENERALE

$A \subset \mathbb{R}^m$, $E \subset \mathbb{R}^N$

$G : A \subset \mathbb{R}^m \longrightarrow E \subset \mathbb{R}^N$ differenziabile

$\underline{x} = (x_1, \dots x_m) \longmapsto G(\underline{x}) = \begin{bmatrix} G_1(\underline{x}) \\ \vdots \\ G_N(\underline{x}) \end{bmatrix}$

$DG(x)$ matrice $N \times m$

$= \left(\dfrac{\partial G_i}{\partial x_j} \right)$ $i = 1 \dots N$, $j = 1 \dots m$

$F : E \subset \mathbb{R}^N \longrightarrow \mathbb{R}^k$

$\underline{y} \in \mathbb{R}^N \longrightarrow \begin{bmatrix} F_1(\underline{y}) \\ \vdots \\ F_k(\underline{y}) \end{bmatrix}$ differenziabile

$DF(y)$ matrice $k \times N$

Allora $F \circ G : A \subset \mathbb{R}^m \longrightarrow \mathbb{R}^k$ è differenziabile in A

$\underline{x} \in A \longmapsto \underline{F}(\underline{G}(\underline{x}))$

e si ha

$$D(F \circ G) = DF(G(x))\, DG(x)$$

$$K \times m \qquad\qquad K \times N \qquad N \times m$$

prodotto di matrici
righe per colonne

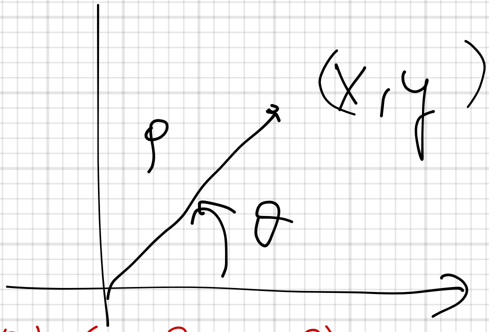
cioè

$$\left(F^i(G(\underline{x}))\right)_{x_j} = \sum_{h=1}^{N} (F^i)_{y_h}(G(\underline{x}))\; G^h_{x_j}(\underline{x}) \qquad \forall i = 1 \dots K \quad \forall j = 1 \dots m$$

$$(AB)_{ij} = \sum_h a_{ih}\, b_{hj}$$

$f(x,y): \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad C^{\cancel{1}\,2}$ ESEMPIO

$z(\rho,\theta) = f(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)$



Vogliamo ottenere le derivate parziali di z a partire da quelle di f

$g(\rho,\theta) = (\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)$

$z(\rho,\theta) = f(g(\rho,\theta))$

$z_\rho(\rho,\theta) =$

$= f_x(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)\cos\theta + f_y(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)\,\mathrm{sen}\,\theta$

$z_\theta(\rho,\theta) = f_x(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)(-\rho\,\mathrm{sen}\,\theta) + f_y(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)\,\rho\cos\theta$

$z_{\rho\theta}(\rho,\theta) = [f_{xx}(\ )(-\rho\,\mathrm{sen}\,\theta) + f_{xy}(\ )\rho\cos\theta]\cos\theta - f_x(\ )\,\mathrm{sen}\,\theta + [f_{yx}(\ )(-\rho\,\mathrm{sen}\,\theta) +$

$+ f_{yy}(\ )\rho\cos\theta]\,\mathrm{sen}\,\theta + f_y(\ )\cos\theta$

$$\Delta f = \nabla^2 f = \sum_{i=1}^{\cancel{N}\,2} f_{x_i x_i} = f_{xx} + f_{yy} \qquad f : \mathbb{R}^{\cancel{N}\,2} \to \mathbb{R}$$

$$\nabla^2 f = \Delta f = 0$$

$$z(\rho, \theta) = f(\rho \cos\theta, \rho \operatorname{sen}\theta)$$

$$\rho = \sqrt{x^2 + y^2} \qquad \theta = \operatorname{arctg} \frac{y}{x}$$

$$f(x,y) = z\left(\sqrt{x^2+y^2}, \operatorname{arctg} \frac{y}{x}\right)$$

$$f_x(x,y) = z_\rho\left(\sqrt{x^2+y^2}, \operatorname{arctg} \frac{y}{x}\right) \frac{x}{\sqrt{x^2+y^2}} + z_\theta(\ldots) \underbrace{\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)}_{= -\frac{y}{x^2+y^2}}$$

$$f_{xx}(\quad) = ? \qquad f_y = ? \qquad f_{yy} =$$

$f : A \subset \mathbb{R}^N \longrightarrow \mathbb{R}$ derivabile "quanto basta", $\underline{x}_0 \in A$

Formula di Taylor del 1° ordine con resto di Peano

$$f(\underline{x}_0+\underline{h}) = f(\underline{x}_0) + \underbrace{\nabla f(\underline{x}_0)\cdot \underline{h}}_{\sum_{i=1}^{N} f_{x_i}(\underline{x}_0)\, h_i} + o(\|\underline{h}\|) \quad \text{per } \underline{h}\to \underline{0}$$

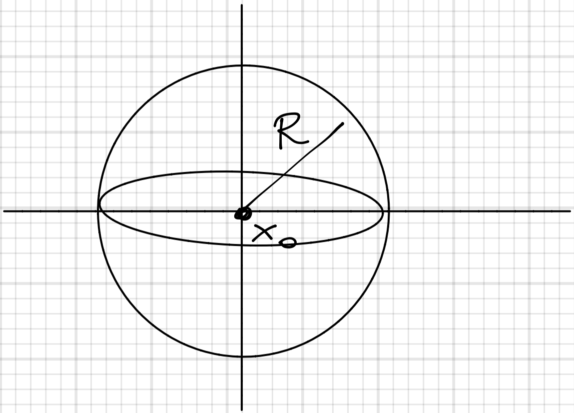
$N=1$

$$f(x_0+h) = f(x_0) + f'(x_0)\,h + o(|h|) \qquad \text{per } h \to 0$$

$$\lim_{\underline{h}\to \underline{0}} \frac{f(\underline{x}_0+\underline{h}) - f(\underline{x}_0) - \nabla f(\underline{x}_0)\cdot \underline{h}}{\|\underline{h}\|} = 0$$

FORMULA DI TAYLOR DEL 1° ORDINE con RESTO DI LAGRANGE

$f \in C^2(B_R(\underline{x}_0))$ $\quad B_R(\underline{x}_0) = \{ x \in \mathbb{R}^N : \|\underline{x} - \underline{x}_0\| < R \}$



In dim 1 $\qquad t \in (0,1)$

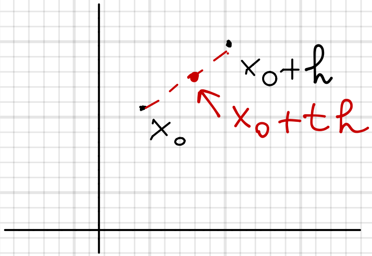
$$f(x_0 + h) = f(x_0) + f'(x_0)\,h + \frac{1}{2} f''(x_0 + th)\,h^2$$

In dim N

$$\underline{v}\cdot\underline{w} = (\underline{v},\underline{w})$$

$$f(\underline{x}_0+\underline{h}) = f(\underline{x}_0) + \nabla f(\underline{x}_0)\cdot\underline{h} + \frac{1}{2}\left(D^2 f(\underline{x}_0+t\underline{h})\,\underline{h}^T, \underline{h}\right)$$

$$D^2 f = \text{matrice hessiana} = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_N} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_N} \\ \cdots & \cdots & \cdots & \cdots \\ f_{x_Nx_1} & f_{x_Nx_2} & \cdots & f_{x_Nx_N} \end{bmatrix}$$



$$(D^2 f\, h, h) = \sum_{i=1}^{N} (D^2 f\, h)_i\, h_i = \sum_i \Big(\sum_j f_{x_i x_j} h_j\Big) h_i =$$

$$= \sum_{i,j} f_{x_i x_j} h_i h_j$$