Es. 1

$$f_X(x,\theta) = \theta(1-\theta)^{x-1} \qquad x = 1,2,3,\dots \qquad \theta \in (0,1)$$

1

$$E(X_i) = \frac{1}{\theta} \qquad V(X_i) = \frac{1-\theta}{\theta^2}$$

1)

$$f_X(x;\theta) = \frac{\theta}{1-\theta}(1-\theta)^x = e^{\log\left\{\frac{\theta}{1-\theta}(1-\theta)^x\right\}} = e^{\left\{x\log(1-\theta) + \log\frac{\theta}{1-\theta}\right\}}$$

$$h(x) = 1 \qquad \eta(\theta) = \log(1-\theta) \qquad T(x) = x \qquad B(\theta) = -\log\frac{\theta}{1-\theta}$$

$S(\underline{X}_n) = \sum_i T(x_i) = \sum_i x_i$ è SUFF. MIN. COMPL.

2)

$$L(\theta;\underline{x}_n) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i-1} = \theta^n(1-\theta)^{\sum x_i - n} = \left(\frac{\theta}{1-\theta}\right)^n (1-\theta)^{\sum x_i}$$

dal th di fattorizzazione otteniamo la stessa statistica suff. della rappresentazione in famiglia esponenziale

$$\ell(\theta;\underline{x}_n) = n\left(\log\theta - \log(1-\theta)\right) + \sum x_i \log(1-\theta) = n\log\theta + \left(\sum x_i - n\right)\log(1-\theta)$$

$$\ell'(\theta;\underline{x}_n) = \frac{n}{\theta} - \frac{\sum x_i - n}{1-\theta} = \frac{n(1-\cancel{\theta}) - (\sum x_i - \cancel{n})\theta}{\theta(1-\theta)} = \frac{n - \theta\sum x_i}{\theta(1-\theta)} \qquad \ell'(\theta;\underline{x}_n) = 0 \iff n - \theta\sum x_i = 0 \Rightarrow \theta^* = \frac{n}{\sum x_i} = \bar{x}^{-1}$$

$$\ell''(\theta;\underline{x}_n) = -\frac{n}{\theta^2} - \frac{\sum x_i - n}{(1-\theta)^2} < 0 \Rightarrow \hat{\theta}_{MV}(\underline{X}_n) = \bar{X}^{-1}$$

3)

$$E(X) = \bar{X} \iff \frac{1}{\theta} = \bar{X} \iff \hat{\theta}_M = \bar{X}^{-1} = \hat{\theta}_{MV}$$

4) Dalle proprietà della media campionaria sappiamo che

$$E(\bar{X}) = E(X) = \frac{1}{\theta} \qquad V(\bar{X}) = \frac{V(X)}{n} = \frac{1-\theta}{n\theta^2} \Rightarrow \hat{\psi} = \bar{X}_n - 1 \text{ è tale che } E(\hat{\psi}) = \frac{1}{\theta} - 1 = \frac{1-\theta}{\theta} \quad \forall\theta\in(0,1) \iff \text{è corretto per } \psi$$

(2)

$V(\hat{\Psi}) = V(\bar{X}) = \frac{1-\theta}{n\theta^2} \Longrightarrow MSE_\theta(\hat{\Psi}) = V(\hat{\Psi}) = \frac{1-\theta}{n\theta^2} \quad \forall\theta \in (0,1)$ ne

$\lim_{n\to\infty} MSE_\theta(\hat{\Psi}) = 0 \quad \forall\theta\in(0,1) \Leftarrow \hat{\Psi}$ è consistente in MSE

(3) Si ... virtù del th di Lehman-Scheffé essendo corretto per Ψ e f.ne della statistica suff. e completa determinata al punto (1)

Es. 2 (1) Dalle proprietà degli stimatori di max verosimiglianza è noto che

$$\hat{\theta}_{MV}(X_n) \stackrel{.}{\sim} N\left(\theta, I(\theta)^{-1}\right)$$

Si tratta dunque di verificare l'espressione dell'info attesa di Fisher

$$I(\theta) = E_\theta\left[I(\theta; x_n)\right] = E_\theta\left[-\ell''(\theta; x_n)\right] = E_\theta\left[\frac{n}{\theta^2} + \frac{\sum X_i - n}{(1-\theta)^2}\right] = \frac{n}{\theta^2} + \frac{1}{(1-\theta)^2}\left(E_\theta\left[\sum X_i\right] - n\right) = \frac{n}{\theta^2} + \frac{1}{(1-\theta)^2}\left(\frac{n}{\theta} - n\right)$$

$$= \frac{n}{\theta^2} + \frac{n}{(1-\theta)^{\cancel{2}}}\frac{\cancel{1-\theta}}{\theta} = \frac{n(1-\theta) + n\theta}{\theta^2(1-\theta)} = \frac{n}{\theta^2(1-\theta)}$$

dunque $\quad \hat{\theta}_{MV}(X_n) \stackrel{.}{\sim} N\left(\theta, \frac{\theta^2(1-\theta)}{n}\right)$

(2) stimando la varianza asintotica con $\frac{\bar{X}^{-2}(1-\bar{X}^{-1})}{n}$, definiamo la seguente quantità pivotale

$$\frac{(\bar{X}^{-1} - \theta)\sqrt{n}}{\bar{X}^{-1}\sqrt{(1-\bar{X}^{-1})}} \stackrel{.}{\sim} N(0,1)$$

da cui discende il seguente $\widehat{IC}_{1-\alpha}(X_n)$ per θ

$$\widehat{IC}_{1-\alpha}(X_n) = \bar{X}^{-1} \pm z_{1-\alpha/2}\, \bar{X}^{-1}\sqrt{\frac{1-\bar{X}^{-1}}{n}}$$

3) $m=20$ $\sum x_i = 30$ $\bar{x} = 1.5$ $\bar{x}^{-1} = 0.667$ $\bar{x}^{-2} = 0.445$ $\widehat{V(\hat{\theta}_{MV})} = \frac{0.445(1-0.667)}{20} = 0.007$

$1-\alpha = 0.90 \Rightarrow 1-\alpha/2 = 0.95$

$$\widehat{IC}_{0.90}(x_{oss}) = \left[0.667 \pm 1.645\sqrt{0.007}\right] = [0.529, 0.805]$$

4)

$$\begin{cases} H_0: \theta = \theta_0 \\ H_1: \theta = \theta_1 \end{cases} \qquad \theta_0 < \theta_1$$

$$\lambda_{01}(x) = \frac{L(\theta_0; x)}{L(\theta_1; x)} = \frac{\left(\frac{\theta_0}{1-\theta_0}\right)^m (1-\theta_0)^x}{\left(\frac{\theta_1}{1-\theta_1}\right)^m (1-\theta_1)^x} = \underbrace{\left(\frac{\theta_0}{\theta_1}\frac{1-\theta_1}{1-\theta_0}\right)^m}_{>0} \cdot \underbrace{\left(\frac{1-\theta_0}{1-\theta_1}\right)^x}_{\text{f.ne crescente di } x \text{ essendo il rapporto } >1} \Longrightarrow \Delta$$

$$\mathcal{R} = \{x \in \mathbb{N} : \lambda_{01}(x) \geq k'\} = \{x \in \mathbb{N} : x \geq k\}$$

5)

$$\alpha = P_{\theta_0}(\mathcal{R}) = P(X < k \mid \theta = \theta_0) = P(X < 3 \mid \theta = \tfrac{1}{3}) = P(X = 1 \mid \theta = 1/3) + P(X = 2 \mid \theta = \tfrac{1}{3}) = \frac{1}{3}\cdot\left(\frac{2}{3}\right)^{1-1} + \frac{1}{3}\left(\frac{2}{3}\right)^{2-1} = \frac{1}{3} + \frac{2}{9} = \frac{5}{9}$$

$\boxed{\text{Es. 3}}$

$X \sim N(\mu = 11.4, \sigma = 1.3)$ $m = 20$

1)

$$\bar{X} \sim N\left(\mu_{\bar{X}} = \mu_X = 11.4,\ \sigma^2_{\bar{X}} = \frac{\sigma^2_X}{m} = \frac{1.3^2}{20} = \frac{1.69}{20} = 0.0845\right) \Longleftrightarrow N(\mu_{\bar{X}} = 11.4, \sigma_{\bar{X}} = 0.291)$$

$$P(9.7 \leq \bar{X} \leq 11) = P\left(\frac{9.7-11.4}{0.291} \leq Z \leq \frac{11-11.4}{0.291}\right) = P(-5.84 \leq Z \leq -1.37) = \Phi(-1.37) - \Phi(-5.84) \simeq \Phi(-1.37) = 0.0901$$

4

(2) $Y = 2S_m^2 - 1$ è trasformazione lineare di S_m^2 che, sotto modello normale, ha la seguente distrib.

$$\frac{(m-1)S_m^2}{\sigma^2} \sim \chi^2_{m-1} \Longrightarrow E\left(\frac{(m-1)S_m^2}{\sigma^2}\right) = m-1 \quad e \quad V\left(\frac{(m-1)S_m^2}{\sigma^2}\right) = 2(m-1) \Longrightarrow$$

$$E(S_m^2) = \sigma^2 \quad e \quad V(S_m^2) = \frac{2\cancel{(m-1)}}{(m-1)^{\cancel{2}}}\sigma^4 \Longrightarrow$$

$$E(Y) = 2E(S_m^2) - 1 = 2\sigma^2 - 1 = 2.38$$

$$V(Y) = 4V(S_m^2) = \frac{8 \cdot 1.69^2}{19} = 1.20$$

(3) $E(\bar{X}_m - S_m^2) = E(\bar{X}) - E(S_m^2) = \mu_X - \sigma_X^2 = 11.4 - 1.69 = 9.71$

$$V(\bar{X}_m - S_m^2) \underset{\substack{\downarrow \\ \text{essendo } \bar{X}_m \perp\!\!\!\perp S_m^2}}{=} V(\bar{X}_m) + V(S_m^2) = 0.0845 + \frac{2 \cdot 1.69^2}{19} = 0.0845 + 0.3006 = 0.3851$$

(4)

$$P(S_m^2 > 2) = P\left(\chi^2_{19} > \frac{19 \cdot 2}{1.69}\right) = P(\chi^2_{19} > 22.49) \simeq 0.25$$

(5)

$$Y = \frac{(\bar{X}_m - \mu)\sqrt{m}}{S_m} \sim T_{m-1} \Longrightarrow F_{T_{19}}(2.093) - F_{T_{19}}(-1.328) = 0.875$$