ESERCITAZIONE 1

(Esercizi capitolo 1 dispense)

1.1 [1] Calcola media (valore atteso) e varianza di $X \sim$ Bernoulliana $(\theta = 0.6)$
Rappresenta graficamente la funzione di massa di probabilità.

$X \sim Bern(\theta) \Longrightarrow \mathcal{X} = \{0,1\}$ = SPAZIO CAMPIONARIO

$\Theta = [0,1]$ = SPAZIO PARAMETRICO

$\mathbb{E}[X] = \theta = 0.6 = 0 \cdot \mathbb{P}(X=0) + 1 \cdot \mathbb{P}(X=1) = \mathbb{P}(X=1) = \theta$

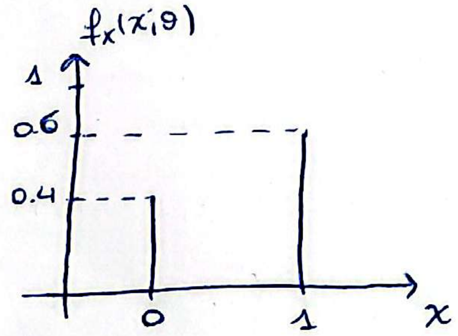
$\mathbb{V}[X] = \theta(1-\theta) = 0.6 \times (1-0.6) = 0.6 \times 0.4 = 0.24$ * FONDO PAGINA

$\mathbb{P}(X=x, \theta) = f_X(x,\theta) = \theta^x (1-\theta)^{1-x}$ con $x = 0,1$

↳ f. massa. prob.

$\mathbb{P}(X=0, \theta=0.6) = 0.6^0 (1-0.6)^{1-0} = 0.4$

$\mathbb{P}(X=1, \theta=0.6) = 0.6^1 (1-0.6)^{1-1} = 0.6$



1.2 [2] Data $X \sim$ Binomiale $(K=12, \theta=0.3)$,
calcola la probabilità che X assuma:

1. Il valore 7
2. Valori compresi tra 7 e 9
3. valori ≥ 2
4. valori < 5 o > 10

Ricordiamo che $X \sim Binomiale(K, \theta) \Longrightarrow \cdot\ \mathcal{X} = \{0, 1, 2, \dots K\}$

$\cdot\ \Theta = [0,1]$

$\cdot\ \mathbb{P}[X = x, \theta] = f_X(x,\theta) = \binom{K}{x} \theta^x (1-\theta)^{K-x}$

$\mathbb{P}[X=7] = \binom{12}{7} 0.3^7 (1-0.3)^{12-7} = \binom{12}{7} 0.3^7 \, 0.7^5$

$\mathbb{P}(7 < X < 9) = \mathbb{P}(X=8) = \binom{12}{8} 0.3^8 \, 0.7^{12-8}$

$\mathbb{P}(X \geq 2) = \sum_{x=2}^{12} \binom{12}{x} 0.3^x \, 0.7^{12-x}$

$\mathbb{P}(X < 5, X > 10) = \sum_{x=0}^{4} \binom{12}{x} 0.3^x \, 0.7^{12-x} + \sum_{x=11}^{12} \binom{12}{x} 0.3^x \, 0.7^{12-x}$

$= 1 - \mathbb{P}(5 \leq X \leq 10) = 1 - \sum_{x=5}^{10} \binom{12}{x} 0.3^x \, 0.7^{12-x}$

$* \mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \Rightarrow \mathbb{E}[X^2] = 0^2 \cdot \mathbb{P}(X=0) + 1^2 \cdot \mathbb{P}(X=1) = \theta$

$= \theta - \theta^2 = \theta(1-\theta)$

1.4 Data X v.a. discreta con $f_X(x;\theta,\alpha) = \dfrac{\Gamma(\alpha+x)}{\Gamma(x+1)\Gamma(\alpha)}\theta^{\alpha}(1-\theta)^{x}$ $\quad x=0,1,2,\dots$

dove $\theta \in (0,1)$ e $\alpha > 0$ costante nota.

a) scrivere modello statistico di base

b) scriverlo anche per il campione casuale $X_1, \dots, X_n$

- $(\mathcal{X}, f_X(\cdot;\theta,\alpha), \Theta) \Longrightarrow (\mathbb{N}, f_X, \Theta = (0,1))$
- $(\mathcal{X}^n, f_n(\cdot,\theta,\alpha), \Theta) \Longrightarrow (\mathbb{N}^n, f_n, \Theta = (0,1))$

dove $f_n(\cdot,\theta,\alpha) = \prod_{i=1}^{n} f_X(x_i,\theta,\alpha) = \prod_{i=1}^{n} \dfrac{\Gamma(\alpha+x_i)}{\Gamma(x_i+1)\Gamma(\alpha)}\theta^{\alpha}(1-\theta)^{x_i}$

$$= \left(\prod_{i=1}^{n} \frac{\Gamma(\alpha+x_i)}{\Gamma(x_i+1)\Gamma(\alpha)}\right)\theta^{n\alpha}(1-\theta)^{\sum x_i}$$

1.6 Data X con densità $f_X(x,\vartheta) = \frac{1}{\vartheta} \mathbb{1}_{[0,\vartheta]}(x)$

a) Scrivere modello statistico di base

b) Determina $\mathbb{E}[X]$ e $\mathbb{V}[X]$

$$\left(\mathcal{X} = [0,\vartheta],\ f_X(x,\vartheta),\ \Theta = \mathbb{R}^+ \right)$$

$$\mathbb{E}[X] = \int_0^\vartheta \frac{x}{\vartheta}\, dx = \frac{1}{\vartheta}\left[\frac{x^2}{2}\right]_0^\vartheta = \frac{\vartheta^2}{2\vartheta} = \frac{\vartheta}{2}$$

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \Longrightarrow \mathbb{E}[X^2] = \int_0^\vartheta \frac{x^2}{\vartheta}\, dx = \left[\frac{x^3}{3\vartheta}\right]_0^\vartheta = \frac{\vartheta^2}{3}$$

$$= \frac{\vartheta^2}{3} - \left|\frac{\vartheta}{2}\right|^2 = \frac{\vartheta^2}{3} - \frac{\vartheta^2}{4} = \frac{4\vartheta^2 - 3\vartheta^2}{12} = \frac{\vartheta^2}{12}$$

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1.B Data X con densità $f_X(x,\vartheta) = \frac{2x}{\vartheta} e^{-x^2/\vartheta} \mathbb{1}_{(0,+\infty)}(x)$; con $\vartheta > 0$

a) Modello statistico per il campione casuale $\underline{X}_n = (X_1 --- X_n)$

$(\mathcal{X}^n = (0,+\infty)^n , f_n(\underline{x}_n, \vartheta) , \Theta = (0,+\infty))$

$$f_n(\underline{x}_n, \vartheta) = \prod_{i=1}^{n} f_X(x_i, \vartheta) = \prod_{i=1}^{n} \left(\frac{2x_i}{\vartheta} e^{-x_i^2/\vartheta} \mathbb{1}_{(0,+\infty)}(x_i) \right) = \left(\frac{2}{\vartheta}\right)^n e^{-\sum_{i=1}^{n} x_i^2/\vartheta} \prod_{i=1}^{n} x_i \mathbb{1}_{(0,+\infty)}(x_i)$$

1.11 $X_1 \dots X_n$ campione casuale t.c. $X_i|\vartheta \sim \text{Poisson}(\vartheta)$, determina la probabilità di osservare il campione $\underline{x}_n = (2,3,1,5,5)$

$$P(\underline{X}_n = \underline{x}_n) = f_n(\underline{x}_n, \vartheta) = \prod_{i=1}^{n} f_X(x_i, \vartheta) = \prod_{i=1}^{n} P(X = x_i) = \prod_{i=1}^{n} \left(\frac{e^{-\vartheta} \vartheta^{x_i}}{x_i!} \right) = \frac{e^{-n\vartheta} \vartheta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

$$= \frac{e^{-5\vartheta} \vartheta^{(2+3+1+5+5)}}{2!3!1!5!5!} = \frac{e^{-5\vartheta} \vartheta^{16}}{2!3!1!5!5!}$$

sostituisco $\vartheta = 2$ e $\vartheta = 3$ e ottengo il risultato.

1.12 $X_1 \dots X_n$ t.c. $X_i|\vartheta \sim \text{Exp}(\vartheta)$ $i=1 \dots n$ iid, calcola f. densità congiunta in corrispondenza del campione $\underline{x}_n = (1.12, 0.88, 0.13, 0.42, 0.36)$.

$$f_n(\underline{x}_n, \vartheta) = \prod_{i=1}^{n} f_X(x_i, \vartheta) = \prod_{i=1}^{n} \left(\frac{1}{\vartheta} e^{-x_i/\vartheta} \right) = \vartheta^{-n} e^{-\sum_{i=1}^{n} x_i/\vartheta} = \vartheta^{-5} e^{-(1.12 + \dots + 0.36)/\vartheta}$$

$$= \vartheta^{-5} e^{-2.91/\vartheta}$$

$$X \sim \text{Exp}(\vartheta) \Rightarrow \begin{cases} \mathcal{X} = \mathbb{R}^+ \\ f_X(x, \vartheta) = \frac{1}{\vartheta} e^{-x/\vartheta} \\ \Theta = \mathbb{R}^+ \end{cases} \qquad \mathbb{E}[X] = \vartheta; \quad V[X] = \vartheta^2$$

$$X \sim \text{EN}(\vartheta) \Rightarrow \begin{cases} \mathcal{X} = \mathbb{R}^+ \\ f_X(x, \vartheta) = \vartheta e^{-\vartheta x} \\ \Theta = \mathbb{R}^+ \end{cases} \qquad \mathbb{E}[X] = \frac{1}{\vartheta}, \quad V[X] = \frac{1}{\vartheta^2}$$