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MACROECONOMICS
SAPIENZA PH.D. IN ECONOMICS

Optimal Policy: LQ Approach

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The New Keynesian micky-mouse model

- **Euler equation** (New Keynesian IS curve with $y_t = c_t$) :

$$c_t = E_t c_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1})$$

- Note (with $y_t = c_t$): Fischer vs. Keynes!
- **Phillips curve** (marginal cost measure, $mc_t = kx_t$).

$$\pi_t = \beta E_t \pi_{t+1} + mc_t$$

- Note, "old style" expectations (Taylor): $E_{t-1}\pi_t$.
- **Forward solutions:**

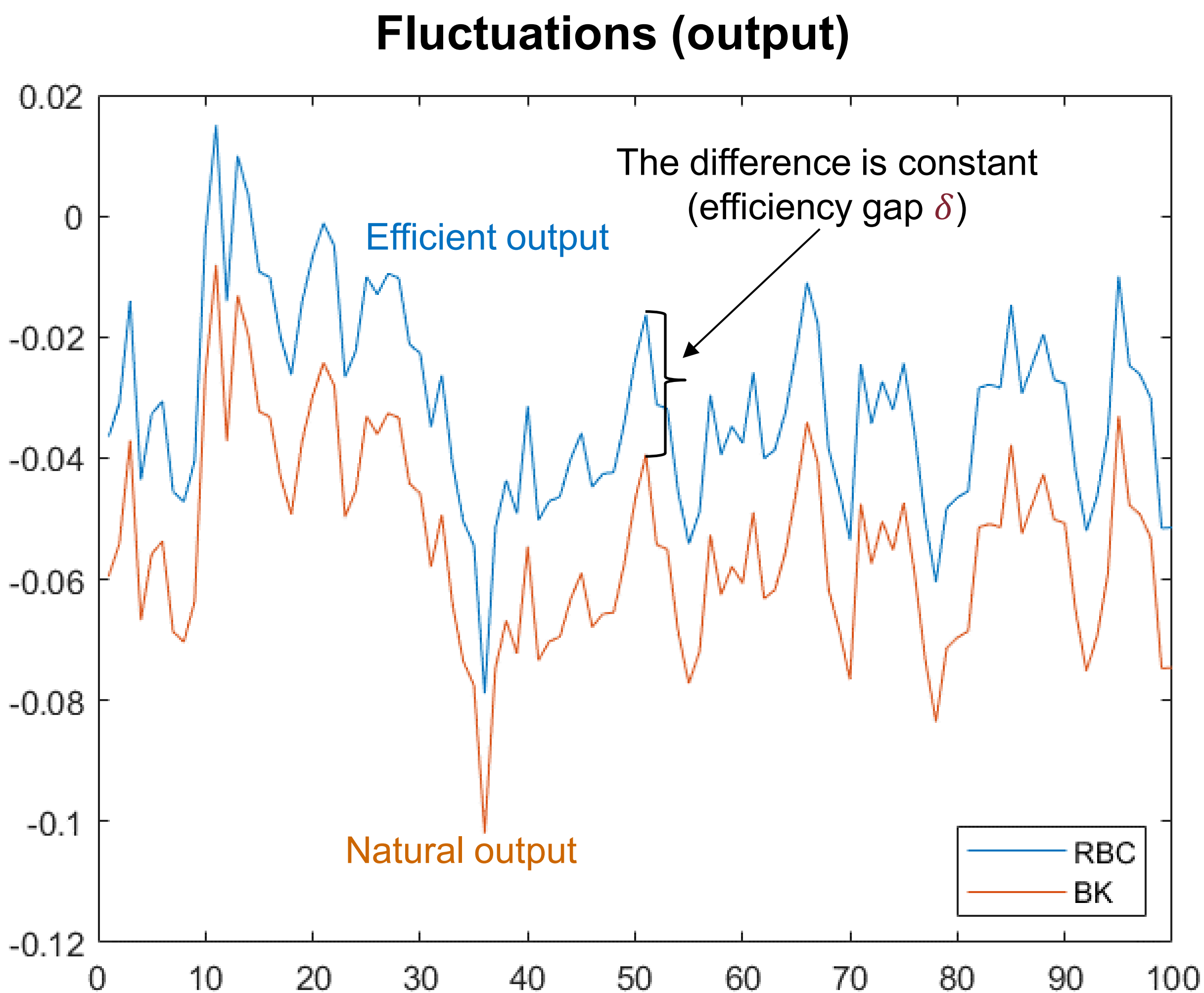
$$c_t = E_t \sum_{j=0}^{\infty}\left[-\frac{1}{\sigma}(i_{t+j} - \pi_{t+1+j}) + g_{t+j}\right] \quad \pi_t = E_t \sum_{j=0}^{\infty} \beta^j mc_t$$

- Intuition?

The time machine

- In 1976, **Lucas** critiqued to policy advice based on aggregate structural models
- In 1982, **Kydland and Prescott** began the Real Business Cycle (RBC) Theory project
  - Micro-foundations and rational expectations in competitive markets
- In 1987, **Blanchard and Kiyotaki** proposed the Monopolistic competition GE model with flexible prices
  - Real rigidities in the otherwise RBC model
- In 1999, **Clarida, Galì, and Gertler** summarized the New Keynesian approach: Monopolistic competition DSGE model with sticky prices (Rotemberg/Calvo)
  - Nominal rigidities in Blanchard and Kiyotaki

Distortions associated with real rigidities



Policy trade-offs and the Phillips Curve

- New Keynesian Phillips curve:

$$\pi_t = \beta E_t \pi_{t+1} + k(y_t - y_t^n)$$

- Criticism: no policy trade-offs (Divine coincidence), a strict inflation targeting is optimal

$$\pi_t = 0 \rightarrow E_t \pi_{t+1} = 0 \rightarrow k(y_t - y_t^n) \rightarrow y_t = y_t^n$$

- Implicit assumption:

$$y_t^e - y_t^n = \delta$$

By lump-sum subsidies: $\delta = 0$

$y_t^n = y_t^e$

- Alternative: time-varying $y_t^e - y_t^n$ gap:

$$\pi_t = \beta E_t \pi_{t+1} + k x_t + u_t$$

- where $x_t = y_t - y_t^n$ and $u_t = k(y_t^e - y_t^n)$

The New Keynesian model

- New Keynesian IS curve and Phillips curve:

$$\begin{cases} x_t = E_t x_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + g_t \\ \pi_t = \beta E_t \pi_{t+1} + k x_t + u_t \end{cases}$$

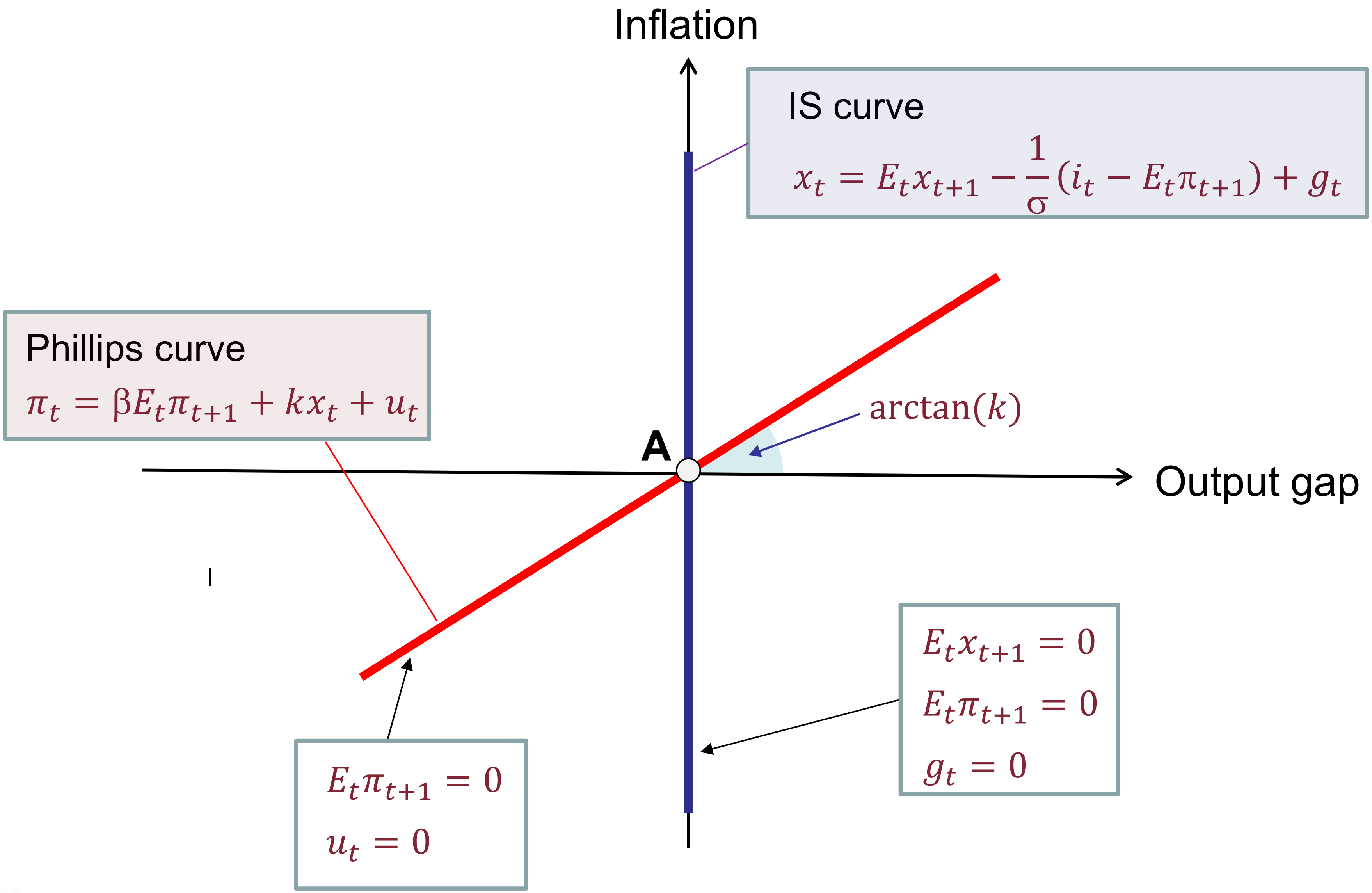
- Endogenous variables g_t and u_t are stochastic processes, e.g.,

$$\begin{cases} g_t = \rho_g g_{t-1} + v_t \\ u_t = \rho_u u_{t-1} + \varepsilon_t \end{cases}$$

- By contrast, v_t and ε_t are exogenous innovations:

$$\begin{cases} v_t \sim N(0, \sigma_v^2) \\ \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \end{cases}$$

Phillips/IS model



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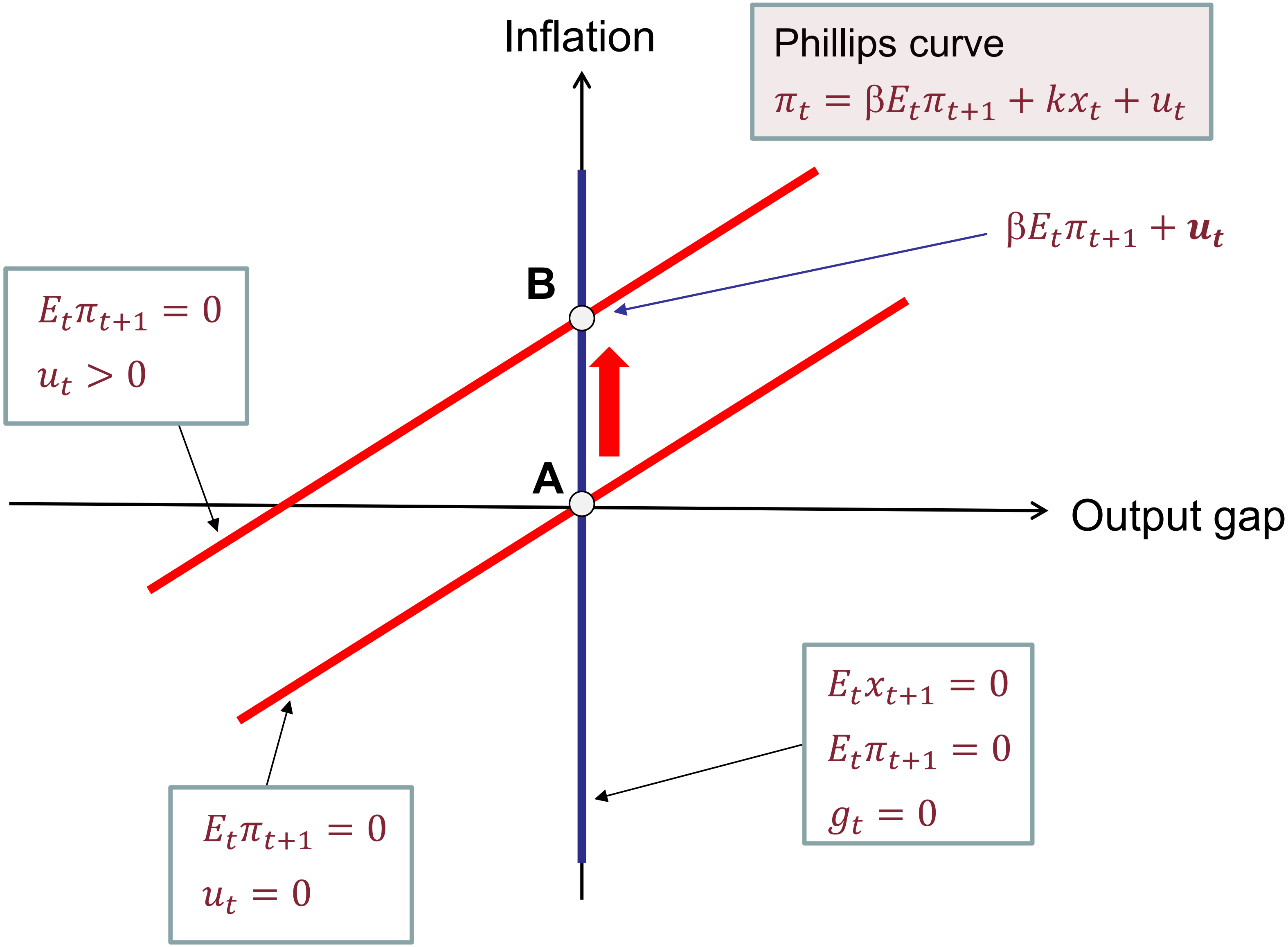
Phillips/IS model: Positive markup shock



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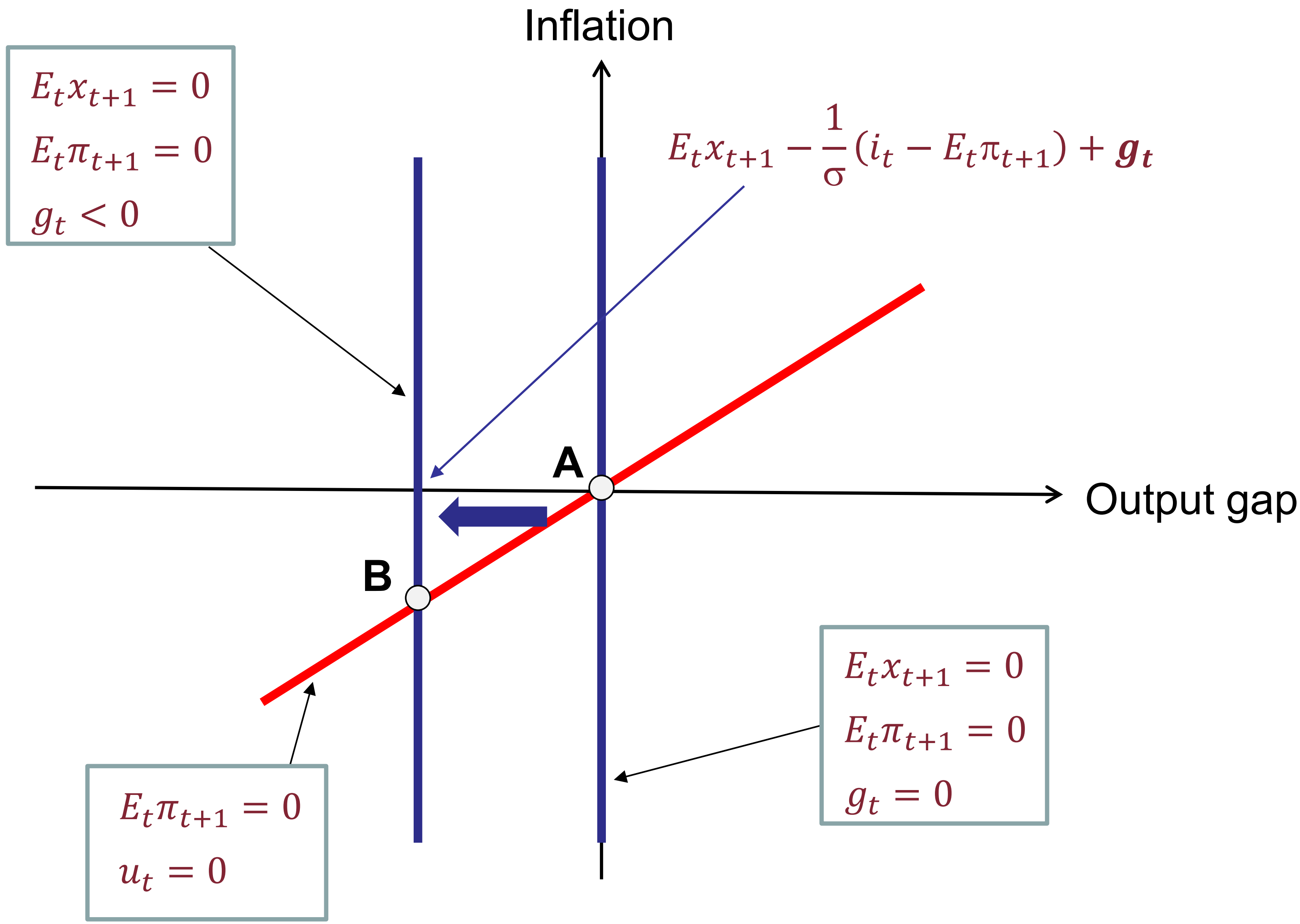
Phillips/IS model: Negative demand shock



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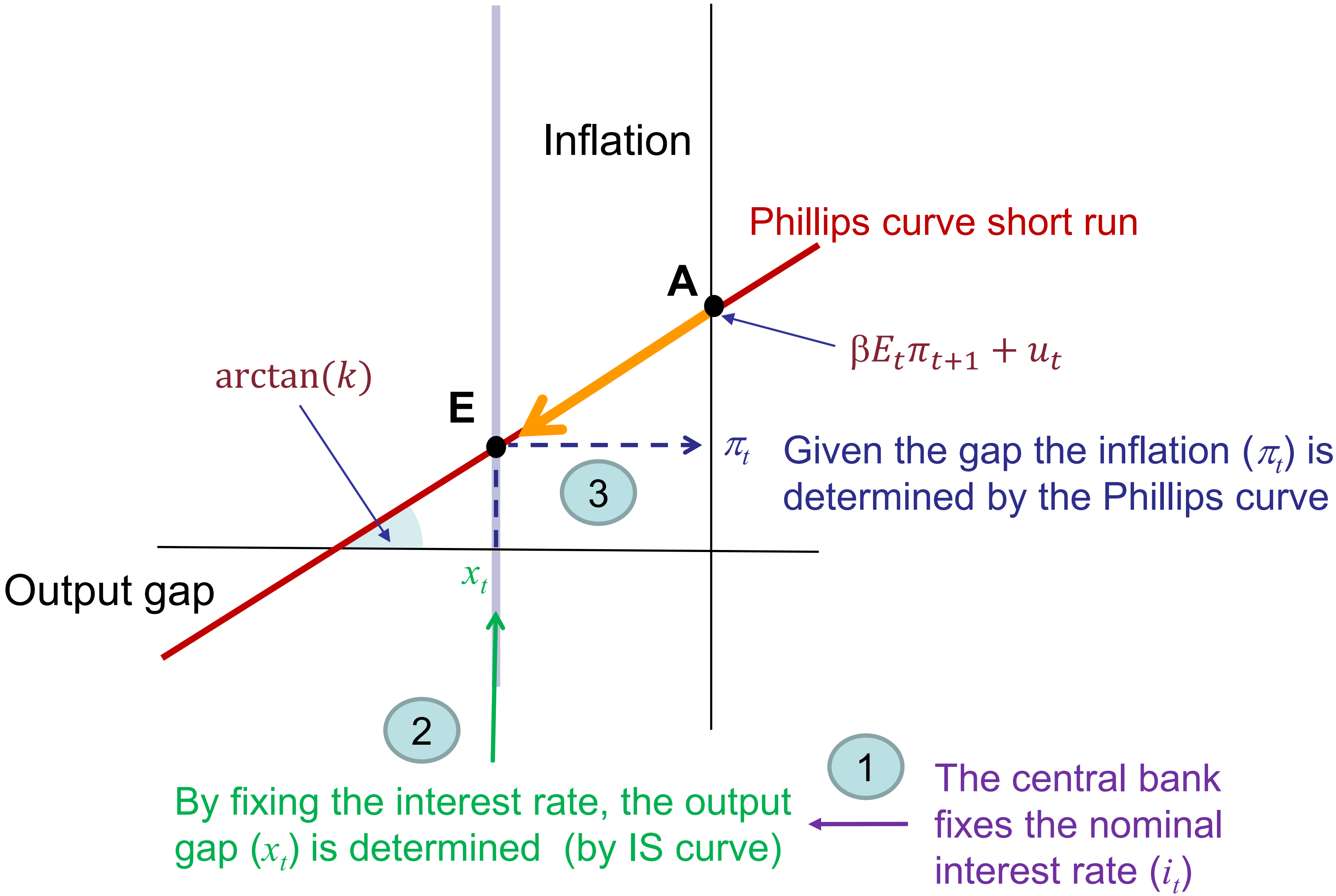
Monetary transmission mechanism



The New Keynesian model

- New Keynesian IS curve and Phillips curve:

$$\begin{cases} x_t = E_t x_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + g_t \\ \pi_t = \beta E_t \pi_{t+1} + k x_t + u_t \end{cases}$$

- where g_t and u_t are stochastic processes, e.g.,

$$\begin{cases} g_t = \rho_g g_{t-1} + v_t, \qquad v_t \sim N(0, \sigma_v^2) \\ u_t = \rho_u u_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \end{cases}$$

- Monetary policy should be defined (r_t^n is fixed)
- **Welfare-based criterion maximization** (optimal policy) alternative to **simple rules** (e.g., Taylor rule)

Welfare-based criterion

- The central bank loss function

$$L = \frac{1}{2} E_t \sum_{i=0}^{\infty} \beta^i (\pi_{t+i}^2 + \alpha_x x_{t+i}^2)$$

- which can be derived as a second-order approximation of the consumer preferences

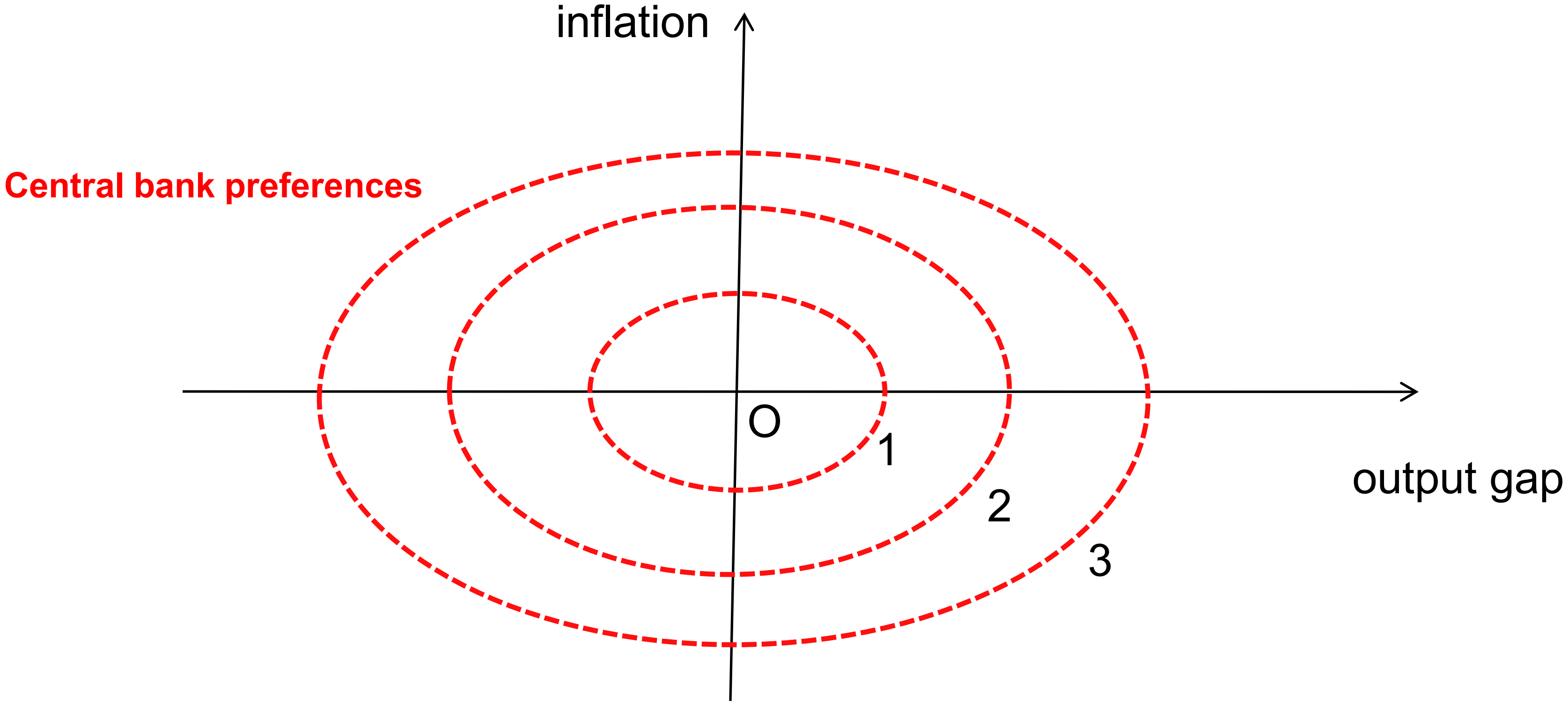
$$\alpha_x = \frac{\kappa}{\epsilon}$$

- where ϵ is the elasticity of substitution between goods in the monopolistic good market
- while κ (slope of the Phillips Curve) increases in
  - the inverse Fisch labor supply elasticity (φ)
  - the labor production coefficient (α)
  - the risk aversion (σ)
  - the price stickiness (b)

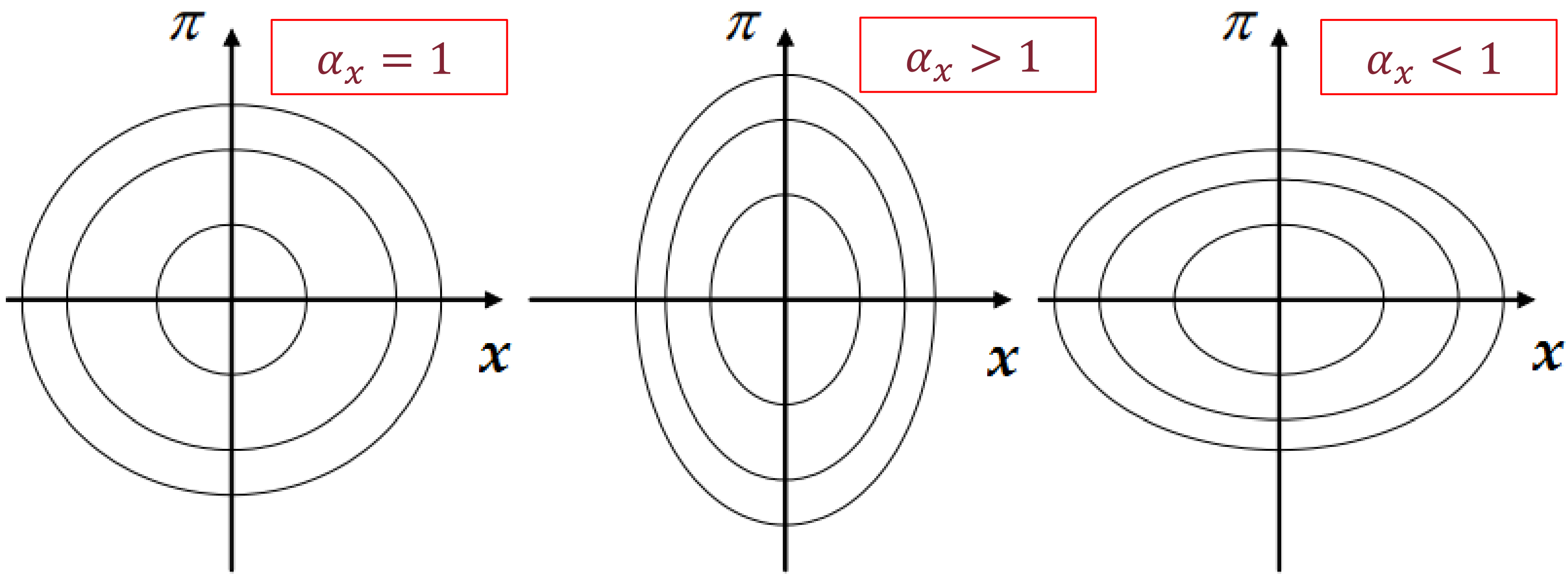
Welfare-based criterion

- The relative costs of output gap fluctuations are increasing in σ, φ, and α.
  – Larger values of those "curvature" parameters amplify the effect of any given deviation of output from its natural level on the size of the gap between the marginal rate of substitution and the marginal product of labor, measuring the economy's aggregate inefficiency.
- The cost of inflation fluctuations are increasing in the market elasticity (ϵ) and the degree of price stickiness (b, $b \to 0$, flat Phillips Curve.)
  – They amplify the degree of price dispersion resulting from any given deviation from zero inflation and the implied misallocation of resources. Some firms produce too much (more labor), others produce too little (less labor.)

Central bank preferences



Alternative preferences



Equal weights

More emphasis on the output gap

More emphasis on inflation

Period loss: $L_t = \pi_t^2 + \alpha_x x_t^2$

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The monetary policy problem

- The central bank preference function

$$\min L = \frac{1}{2}(\pi_t^2 + \alpha_x x_t^2) + \frac{1}{2} E_t \sum_{i=1}^{\infty} \beta^i (\pi_{t+i}^2 + \alpha_x x_{t+i}^2)$$

- Subject to

$$\begin{cases} \pi_t = \beta E_t \pi_{t+1} + k x_t + u_t \\ \pi_{t+1} = \beta E_{t+1} \pi_{t+2} + k x_{t+1} + u_{t+1} \\ \qquad \qquad \qquad \cdots \end{cases}$$

where $u_t = \rho_u u_{t-1} + \varepsilon_t, \;\; \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$

$$\begin{cases} x_t = E_t x_{t+1} - \sigma^{-1}(i_t - E_t \pi_{t+1}) + g_t \\ x_{t+1} = E_{t+1} x_{t+2} - \sigma^{-1}(i_{t+1} - E_{t+1} \pi_{t+2}) + g_{t+1} \\ \qquad \qquad \qquad \cdots \end{cases}$$

where $g_t = \rho_g g_{t-1} + v_t, \;\; v_t \sim N(0, \sigma_v^2)$

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The two-step central bank's problem

- **Step one: Optimal policy**
  - Find the optimal **combination of inflation and output gap** (path) that minimize the social loss given the constraint (Phillips curve)
- **Step two: Implementation problem**
  - Find the **nominal interest rate** (path) that implement the optimal policy
- In our case the two steps can be separate, in other cases it is not possible! (e.g., financial friction costs, cost channel …)

Monetary policy regimes

- Discretion: The CB reoptimize each period
  - Agents anticipate CB's choices. As a result, in each period, the central bank takes expectations are gives, when it minimizes its loss
- Delegation/targeting: The CB operates under discretion, but it is credibly constrained to minimize a given loss (e.g., $L_t = \pi_t^2$, i.e., pure inflation targeting)
  - Agents account for the CB's loss in anticipating its actions: different losses lead to different expectations
- Commitment: The CB credibly commits to set the monetary policy according to a given (feedback) rule
  - Agents account for the CB's loss in forecasting: different rules lead to different expectation paths

Optimal discretionary policy

- Each period the Central Bank chooses (π_t, x_t) to

$$\min L = \frac{1}{2}(\pi_t^2 + \alpha_x x_t^2) + L_0$$

subject to $\pi_t = kx_t + v_t$, where $v_t = \beta E_t \pi_{t+1} + u_t$

- Optimality condition:

$$\pi_t = -\frac{\alpha_x}{k} x_t$$

- "Lean against the wind policy." If inflation is above target, contract demand is below capacity by raising the interest rate, and vice versa when it is below target
- How much the central bank should reduce x_t depends positively on the gain in reducing inflation per unit of output loss, k, (marginal benefit) and inversely on the relative weight on output losses, α_x (marginal cost)

Optimal discretionary policy

- Each period the Central Bank chooses (π_t, x_t) to

$$\min L = \frac{1}{2}\left({\pi_t}^2 + \alpha_x {x_t}^2\right) + L_0$$

- Subject to $\pi_t = kx_t + v_t$, where $v_t = \beta E_t \pi_{t+1} + u_t$
- Optimality condition:

$$\boxed{\pi_t = -\frac{\alpha_x}{k} x_t}$$

- Equilibrium

$$\boxed{\begin{aligned} x_t &= -k\psi u_t \\ \pi_t &= \alpha_x \psi u_t \end{aligned}}$$

where $\psi = k^2 + \alpha_x(1 - \beta\rho_u)$.

- Equilibrium is obtained by combining the optimality condition with the Phillips curve (solving forward)

Discretionary equilibrium: Solution

- Three-equation system

$$\begin{cases} (1) \quad \pi_t = \beta E_t \pi_{t+1} + k x_t + u_t \\ (2) \quad \pi_t = -\frac{\alpha_x}{k} x_t \\ (3) \quad u_t = \rho_u u_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \end{cases}$$

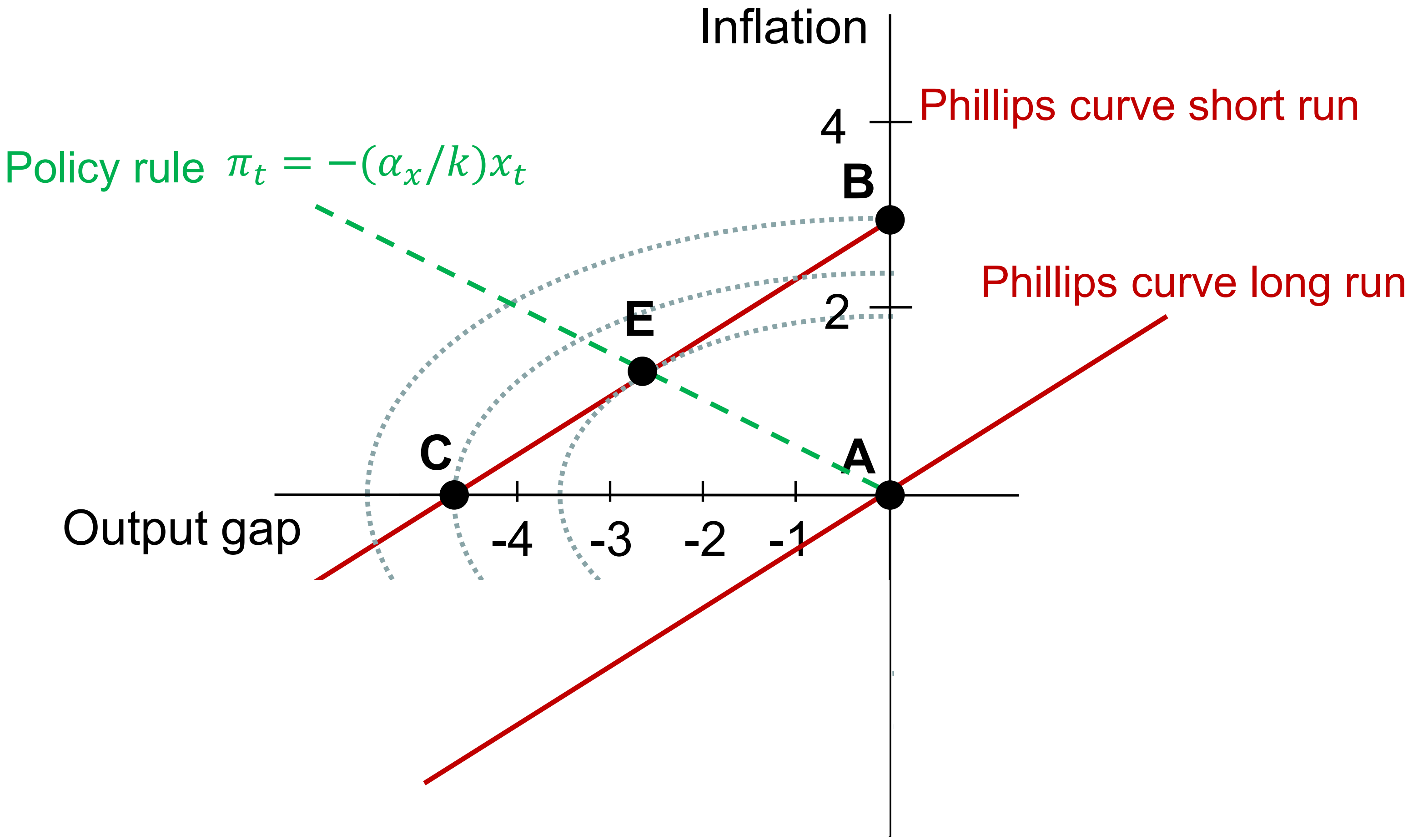
- It follows from (1) and (2)

$$\pi_t = \beta E_t \pi_{t+1} - \frac{k^2}{\alpha_x} \pi_t + u_t \Rightarrow \pi_t = \frac{\alpha_x}{\alpha_x + k^2} (\beta E_t \pi_{t+1} + u_t)$$

- So far iterating forward and using (3) [$E_t u_{t+j} = \rho_u^j u_t$]:

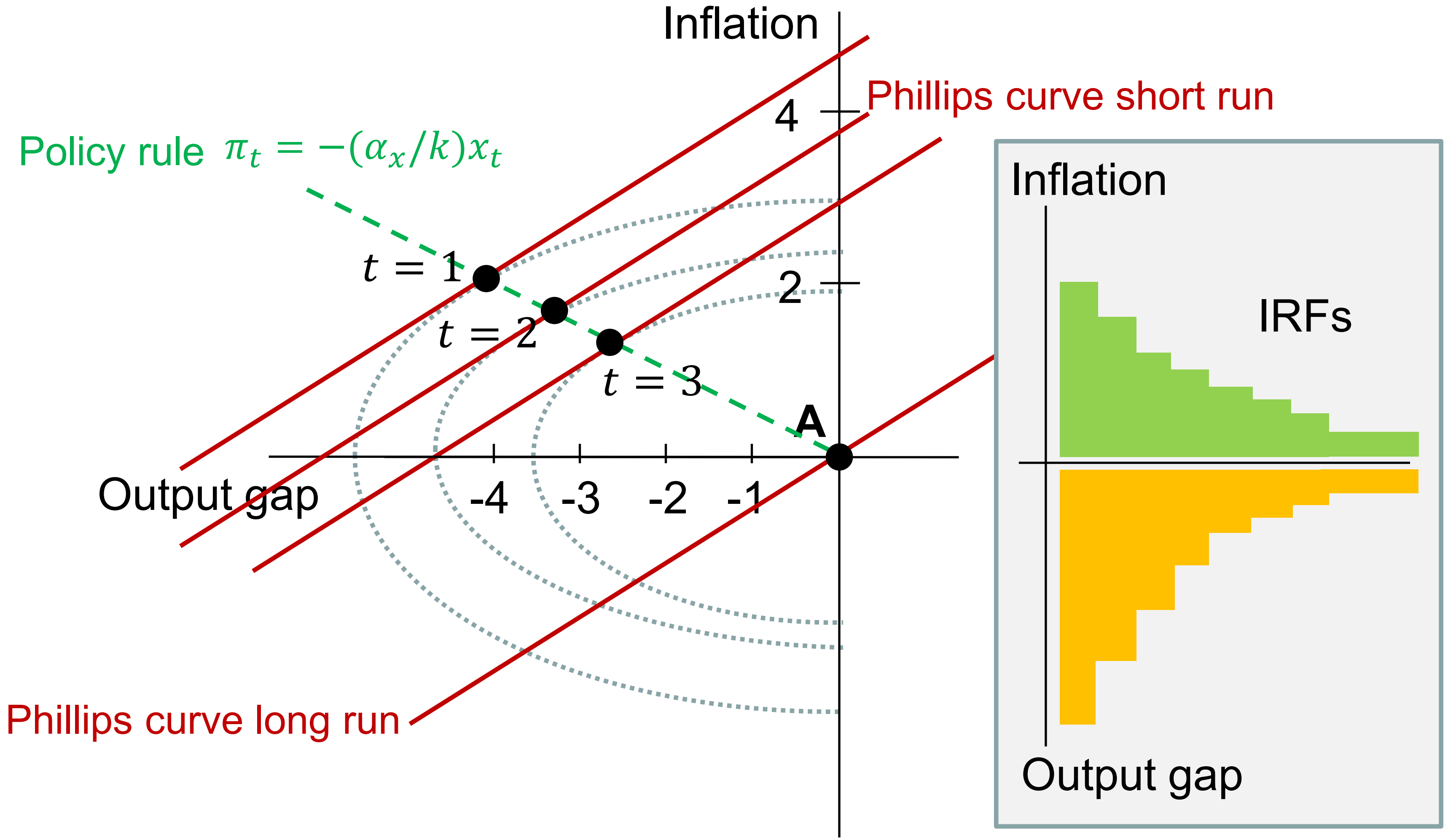
$$\begin{cases} \pi_t = \dfrac{\alpha_x E_t \sum_{j=0}^{\infty} \left(\frac{\alpha_x \beta}{\alpha_x + k^2}\right)^j u_{t+j}}{\alpha_x + k} = \dfrac{\alpha_x \sum_{j=0}^{\infty} \left(\frac{\alpha_x \beta \rho_u}{\alpha_x + k^2}\right)^j u_t}{\alpha_x + k} = \dfrac{\alpha_x}{\alpha_x (1 - \beta \rho_u) + k^2} u_t \\ x_t = -\dfrac{\alpha_x}{k} x_t = -\dfrac{k}{\alpha_x} \dfrac{\alpha_x}{\alpha_x (1 - \beta \rho_u) + k^2} u_t = -\dfrac{k}{\alpha_x (1 - \beta \rho_u) + k^2} u_t \end{cases}$$

Optimal policy (discretion)



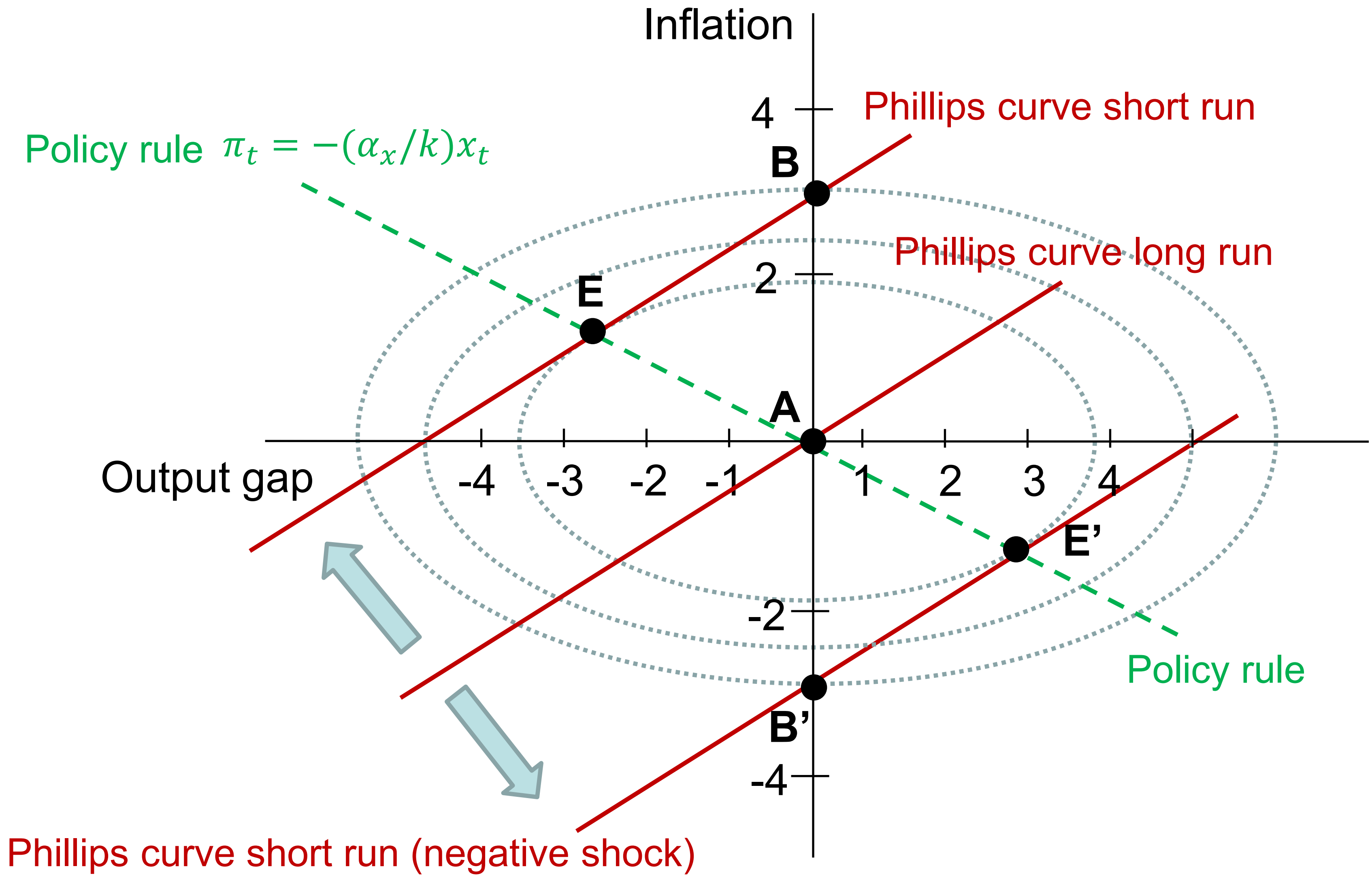
Assuming no persistence: a white noise shock, so $E_t\varepsilon_{t+1} = E_t\pi_{t+1} = 0$, then the upward shift of the curve is $v_t = \varepsilon_t$

Optimal policy (discretion): IRFs



Assuming persistence: $E_t u_{t+1} = \rho_u \varepsilon_t$ and $E_t \pi_{t+1} = \alpha_x \psi \rho_u \varepsilon_t$, so $v_t = (1 + \alpha_x \psi \rho_u)\varepsilon_t$

Shocks can be either positive or negative



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The Phillips trade-off

- Optimal policy rule:

$$\pi_t = -\frac{\alpha_x}{k} x_t$$

- Equilibrium conditional to the shock u_t, implies a trade off between stabilizing the output gap or inflation:

$$x_t = -k\psi u_t \text{ and } \pi_t = \alpha_x \psi u_t$$

- Shocks can be positive or negative: unconditional means are zero, but:

$$var(x_t) = (k\psi)^2 \sigma_u^2$$
$$var(\pi_t) = (\alpha_x \psi)^2 \sigma_u^2$$

- The trade off is on variabilities.

$$\frac{\partial var(x_t)}{\partial \alpha_x} < 0 \qquad \frac{\partial var(\pi_t)}{\partial \alpha_x} > 0$$

The Phillips trade-off: Add-in

- Remember that $u_t = \rho_u u_{t-1} + \varepsilon_t$ with $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$, then:

$$E(u_t) = E(\varepsilon_t) = 0$$

$$var(u_t) = \sigma_u^2 = \frac{\sigma_\varepsilon^2}{1-\rho_u^2}$$

- Equilibrium conditional to the shock u_t, implies:

$$x_t = -k\psi u_t \text{ and } \pi_t = \alpha_x \psi u_t$$

- Shocks are positive or negative, unconditional means:

$$E_t(x_t) = -k\psi E_t(u_t) = 0$$
$$E_t(\pi_t) = \alpha_x \psi E_t(u_t) = 0$$

- Unconditional variances:

$$var(x_t) = E_t(x_t^2) - E_t(x_t)^2 = (k\psi)^2 E_t(u_t^2) = (k\psi)^2 \sigma_u^2$$
$$var(\pi_t) = E_t(\pi_t^2) - E_t(\pi_t)^2 = (\alpha_x\psi)^2 E_t(u_t^2) = (\alpha_x\psi)^2 \sigma_u^2$$

Interest rate rule

- At the equilibrium, the central bank desires

$$x_t = -k\psi u_t \qquad \pi_t = \alpha_x \psi u_t$$

- By IS curve: $x_t = E_t x_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1} - r_t^n) + g_t$:

$$-k\psi u_t = -k\psi E_t u_{t+1} - \frac{1}{\sigma}(i_t - \alpha_x \psi E_t u_{t+1} - r_t^n) + g_t$$

- Solving by i_t (remember $E_t u_{t+1} = \rho_u u_t$):

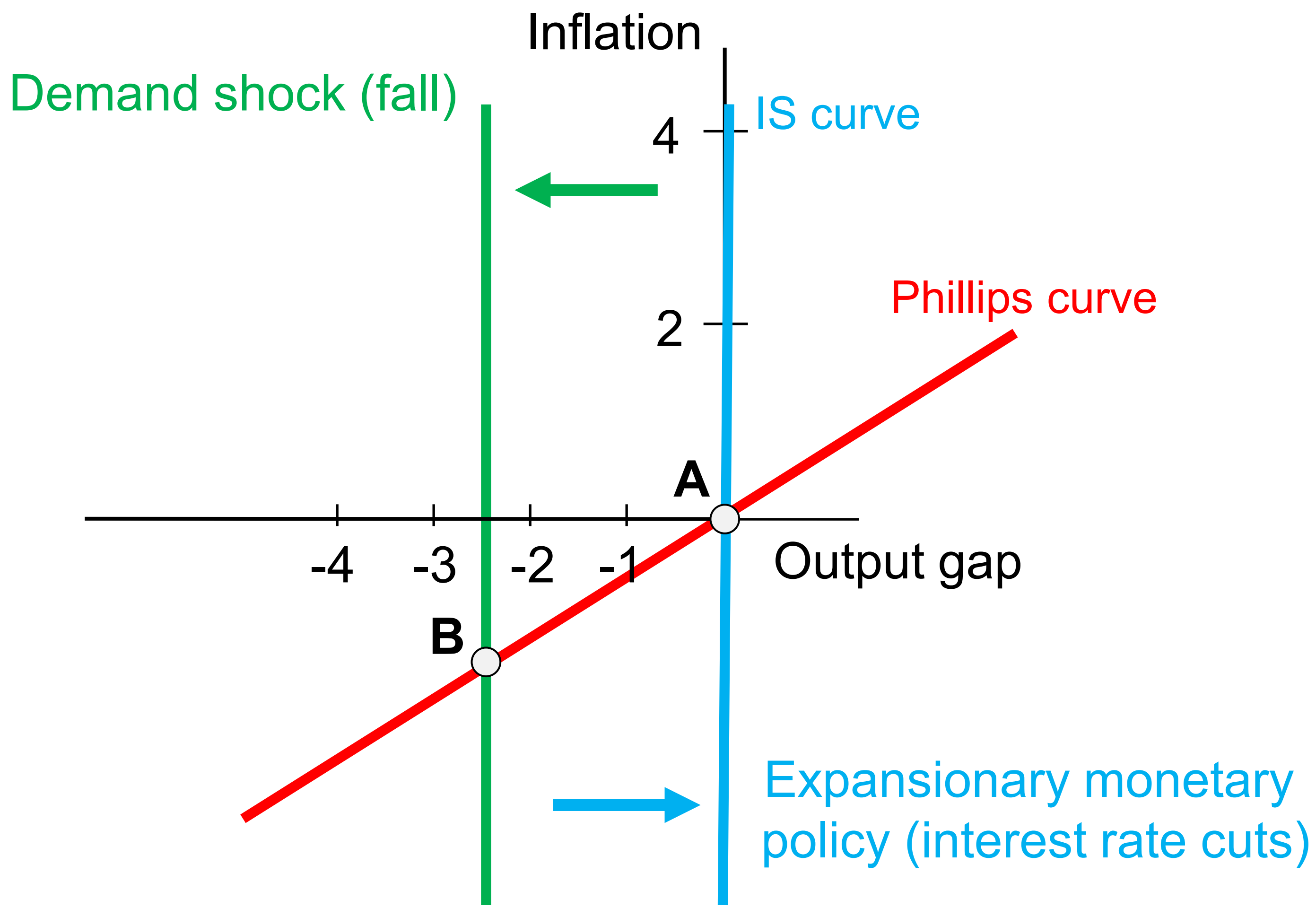
$$i_t = \gamma_\pi \rho_u \alpha_x \psi u_t + \sigma g_t + r_t^n$$

- where $\boxed{\gamma_\pi = 1 + \frac{k(1+\rho_u)}{\rho_u \alpha_x \psi} > 1}$
- Since $\rho_u \alpha_x \psi u_t = E_t \pi_{t+1}$:

$$\boxed{i_t = \gamma_\pi E_t \pi_{t+1} + \sigma g_t + r_t^n}$$

- What can you say about determinacy?

What about demand shocks?



Result summary: Discretion

- **Fact A (trade-off in variances):** To the extent cost push inflation is present, there exists a short run trade-off between inflation and output volatilities.
- **Fact B (Taylor policy and REE stability):** Under the optimal policy, in response to a rise in expected inflation, nominal rates should rise sufficiently to increase real rates, assuring REE stability.
- **Fact C (demand stabilization is a free lunch):** If there is a demand shock, the central bank react by varying the interest rate of σg_t, this in turn offsets the shock that has no effect on the output gap (and thus on inflation).
- But, what about if the ZLB is binding?

Optimal policy with commitment

- State-contingent policy $\{\pi_i, x_i\}_t^{\infty}$ that

$$\min L = \frac{1}{2} E_t \sum_{t=i}^{\infty} \beta^i (\pi_{t+i}^2 + \alpha_x x_{t+i}^2)$$

- Subject to the sequence of constraints:

$$\pi_t = \beta E_t \pi_{t+1} + k x_t + u_t$$

- Lagrange (maximize):

$$-E_t \sum_{i=0}^{\infty} \beta^i \left(\frac{\pi_{t+i}^2 + \alpha_x x_{t+i}^2}{2} + \gamma_{t+i} (\pi_{t+i} - \beta \pi_{t+i+1} - k x_{t+i} - u_{t+i}) \right)$$

- First order conditions (starting from $t = 0$)

$$\begin{cases} \pi_0 + \gamma_0 = 0 & \\ \pi_t + \gamma_t - \gamma_{t-1} = 0 & \text{for } t = 1,2,3 \ldots \\ \alpha_x x_t - k\gamma_t = 0 & \text{for } t = 0,1,2, \ldots \end{cases}$$

Commitment's time inconsistency

- The central bank commits to behave according its first-order conditions (starting from $t = 0$)

$$\pi_0 + \gamma_0 = 0$$

$$\boxed{\pi_1 + \gamma_1 - \gamma_0 = 0}$$

$$\pi_t + \gamma_t - \gamma_{t-1} = 0 \qquad \text{for } t = 2,3 \dots$$

$$\alpha_x x_t - k\gamma_t = 0 \qquad \text{for } t = 0,1,2, \dots$$

- Reoptimizing at $t = 1$, we obtain

$$\boxed{\pi_1 + \gamma_1 = 0}$$

$$\pi_t + \gamma_t - \gamma_{t-1} = 0 \qquad \text{for } t = 2,3 \dots$$

$$\alpha_x x_t - k\gamma_t = 0 \qquad \text{for } t = 0,1,2, \dots$$

Timeless perspective

- We assume a timeless perspective.
- An optimal state-contingent policy rule is derived from an optimization process, but the initial condition are ignored (they were implemented far in the past).
- The initial conditions are different from those operating in the other periods since these do not internalize the rule's effects on expectations, which are initially given when the central bank implements the rule.
- Note that this does make
  - neither the central bank policy time consistent
  - nor globally optimal

Timeless perspective

- First-order conditions:

$$\alpha_x x_t - k\gamma_t = 0$$

$$\pi_t + \gamma_t - \gamma_{t-1} = 0$$

- For $t = 0,1,2,\ldots$ and where $\gamma_{-1} = 0$.
- Therefore, at zero (far in the past): $\pi_0 = -\gamma_0$, while at $t > 0$, the central bank implements: $\pi_t + \gamma_t - \gamma_{t-1} = 0$.
- Ignore the focs at time zero, then

$$\left.\begin{array}{l} \alpha_x x_t - k\gamma_t = 0 \longrightarrow \gamma_t = \dfrac{\alpha_x}{k} x_t \\ \alpha_x x_{t-1} - k\gamma_{t-1} = 0 \rightarrow \gamma_{t-1} = \dfrac{\alpha_x}{k} x_{t-1} \end{array}\right\} \pi_t + \gamma_t - \gamma_{t-1} = 0$$

- Combining, we get the optimal policy rule:

$$\boxed{\pi_t = -\frac{\alpha_x}{k}(x_t - x_{t-1})}$$

Solution: Rewriting the optimal policy rule

- Solution is obtained from the optimal policy rule:

$$x_t = x_{t-1} - \frac{k}{\alpha_x}\pi_t \Rightarrow x_t = -\frac{k}{\alpha_x}p_t$$

[see (later) the slides on the price targeting rule]

- Phillips curve: $p_t - p_{t-1} = \beta E_t p_{t+1} - \beta p_t + k x_t + u_t$, so

$$p_t - p_{t-1} = \beta E_t p_{t+1} - \beta p_t - \frac{k^2}{\alpha_x}, p_t + u_t$$

$$\left(1 + \beta + \frac{k^2}{\alpha_x}\right) p_t = \beta E_t p_{t+1} + p_{t-1} + u_t$$

$$\boxed{p_t = \beta\gamma_C E_t p_{t+1} + \gamma_C (p_{t-1} + u_t)}$$

- where $\gamma_C = \frac{1}{1+\beta+\frac{k^2}{\alpha_x}} \in (0,1)$

Solution: p_t, x_t, and π_t

- We need to solve:

$$p_t = \beta\gamma_C E_t p_{t+1} + \gamma_C(p_{t-1} + u_t)$$

- Guess the solution is

$$\boxed{p_t = \psi_p p_{t-1} + \psi_u u_t}$$

- By using method undetermined coefficients, we find

$$\boxed{\psi_p = \frac{1-\sqrt{1-4\beta\gamma_C^2}}{2\beta\gamma_C}} \quad \text{and} \quad \boxed{\psi_u = \frac{\psi_p}{1-\rho_u\psi_p}}$$

- Given the solution for p_t, then

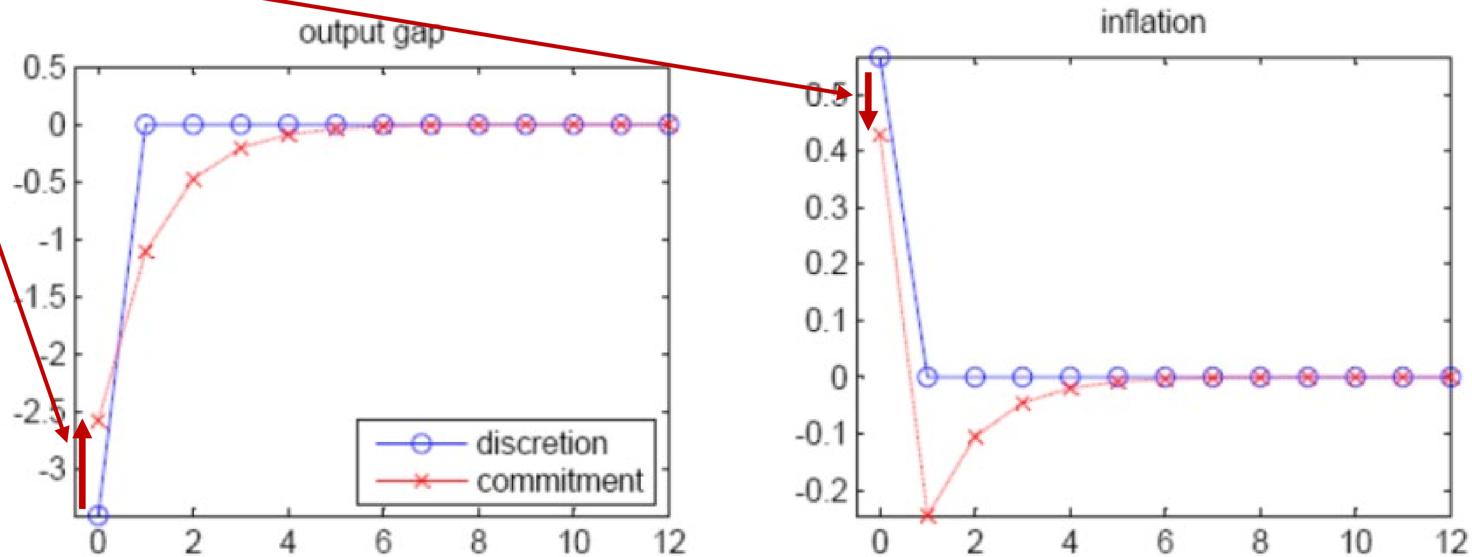
$$x_t = -\frac{k}{\alpha_x} p_t = -\frac{k\psi_p}{\alpha_x} p_{t-1} + -\frac{k\psi_u}{\alpha_x} u_t$$

$$\pi_t = p_t - p_{t-1} = (\psi_p - 1)p_{t-1} + \psi_u u_t$$

Commitment versus discretion

The stabilization bias

Current trade-off improvement



Commitment versus discretion

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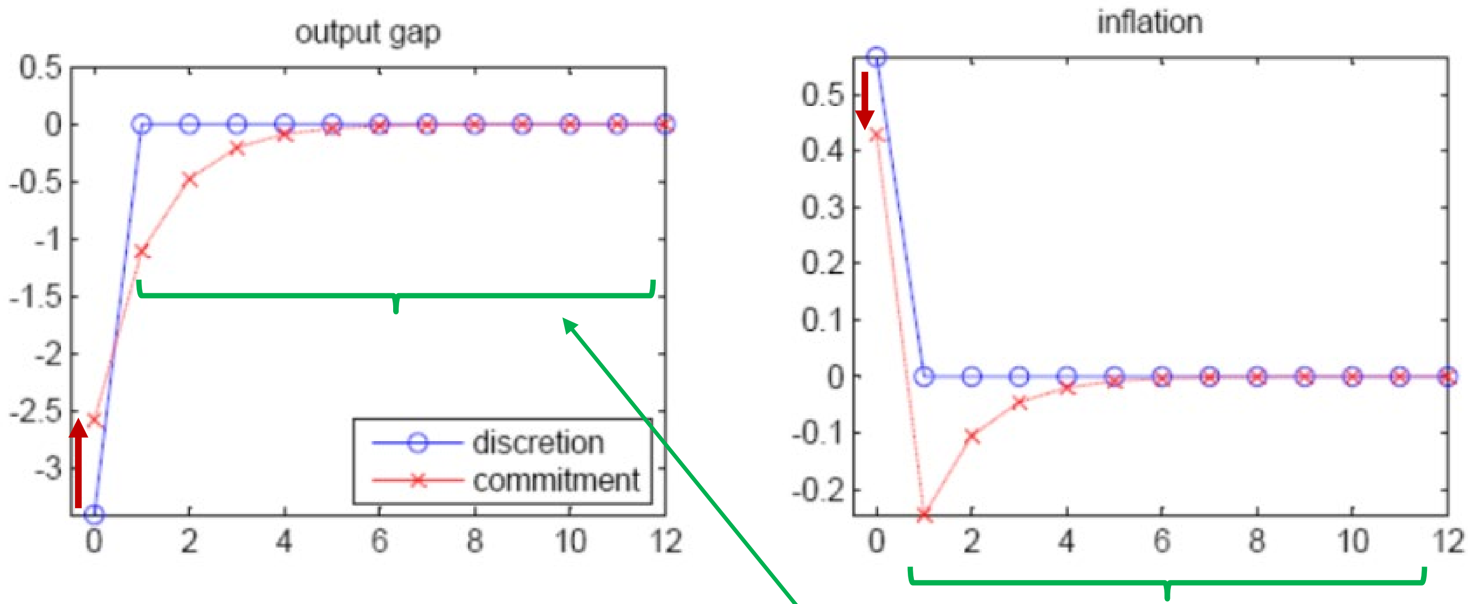


costly artificial persistence

Result summary: Timeless perspective

- **Fact D (timeless perspective):** Although zero output gap outcome is feasible once a positive (negative) noise shock is vanished, under timeless perspective commitment, the central bank find optimal to maintain a persistently negative (positive) output gap and inflation.
  - Commitment creates a costly artificial persistence, but it also lows expectations of future inflation improving the current trade-off between inflation and output gap.
  - It implies a current improvement at the cost of future needs of stabilization; however, commitment dominates discretion, because the convexity of the loss and discounting.

Price targeting rule

- The optimal policy rule is

$$\pi_t = -\frac{\alpha_x}{k}(x_t - x_{t-1})$$

- Alternative representation is based on price targeting rule [p_{-1} target]):

$$\boxed{x_t = -\frac{k}{\alpha_x}(p_t - p_{-1})}$$

- For $t = 0,1,2,\ldots$
- Note that p_{-1} is not p_{t-1}.
- "Lean against the wind policy." If **price** is above target (initial price level, p_{-1}), contract demand is below capacity by raising the interest rate, and vice versa

Price targeting derivation

- Solution is obtained from the optimal policy rule:

$$\pi_t = -\frac{\alpha_x}{k}\Delta x_t \Rightarrow x_t - x_{t-1} = -\frac{k}{\alpha_x}\pi_t$$

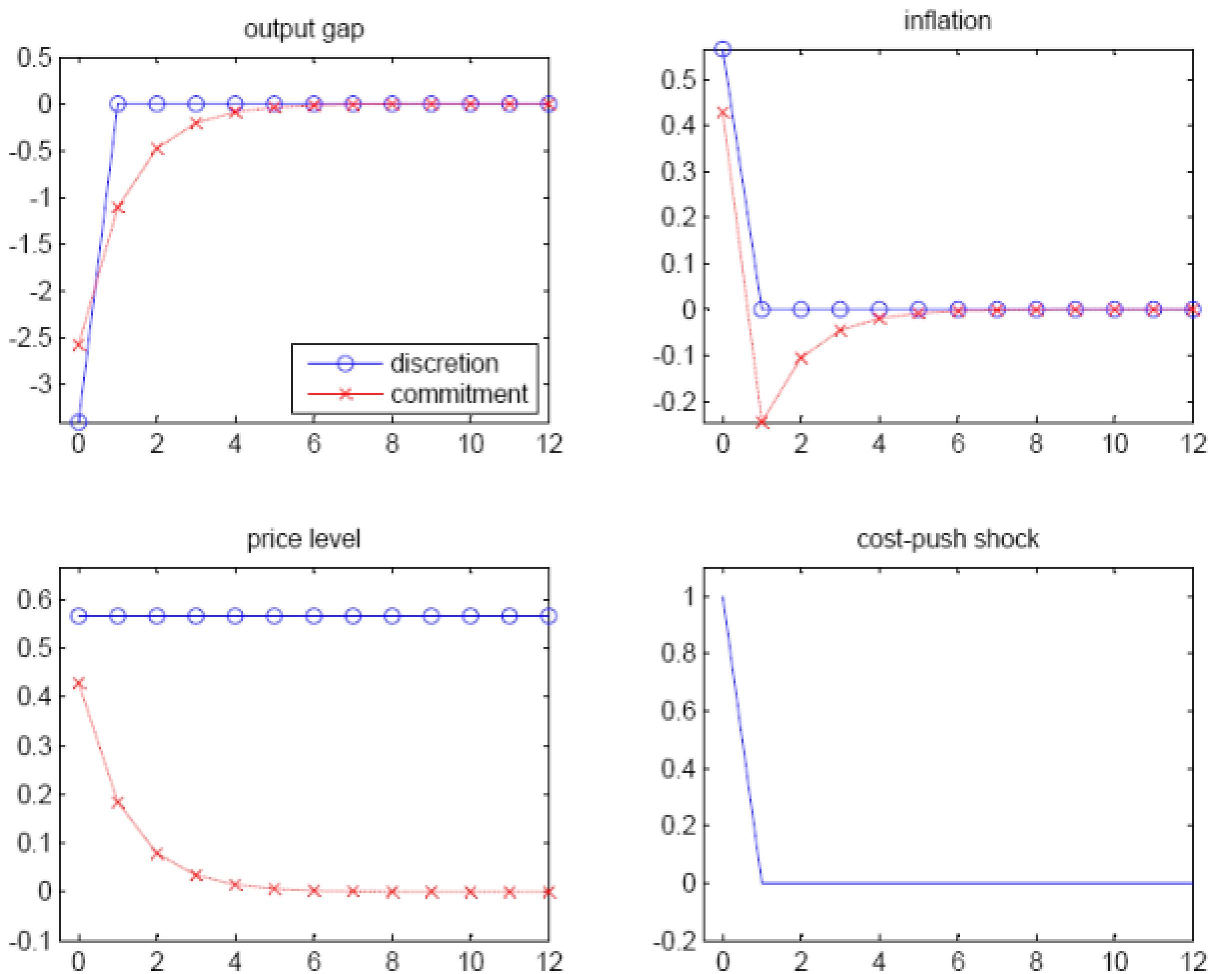
- Solving

$$x_t = -\frac{k}{\alpha_x}\sum_0^t \pi_i$$

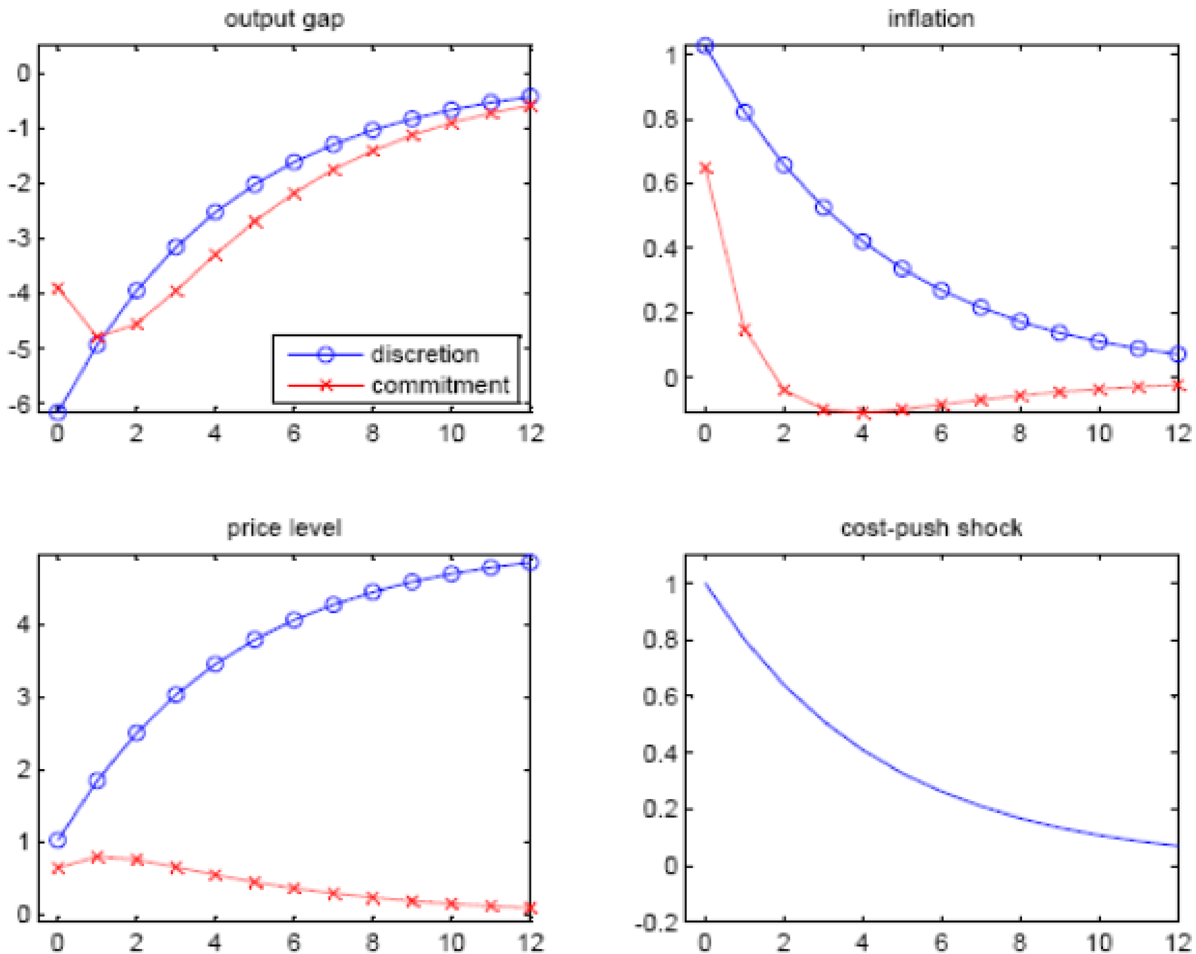
- As $p_t = p_{-1} + \sum_0^t \pi_i$, then the optimal policy:

$$x_t = -\frac{k}{\alpha_x}(p_t - p_{-1})$$

Transitory cost-push shock ($\rho_u = 0$)



Permanent cost-push shock ($\rho_u \in (0,1)$)



Result summary: Further considerations

- **Fact E (price targeting rule):** Timeless perspective solution corresponds to a price targeting rule.
- **Fact F (timeless perspective optimality):** Timeless perspective commitment dominates discretion, but it is not an optimal policy (one first-order condition is ignored). Therefore. There may exist other state-contingent feedback rules that dominate timeless perspective.
- **Fact G (global or full commitment):** Accounting for the initial condition (global or full commitment) leads to a policy rule that implies REE indeterminacy. [proof not reported here, see CGG (1999)].