Sostituzioni speciali.

4. Integrali del tipo $\displaystyle\int R\left(x, \underbrace{\sqrt{1-x^2}}_{-1 \le x \le 1}\right) dx$

Varie sostituzioni possibili.

a) $x = \cos t$ (possibile in quanto $x \in [-1,1]$)
$t \in [0,\pi]$.

$$\sqrt{1-x^2} = \sqrt{1-\cos^2 t} = |\sin t| = \sin t$$

perché $t \in [0,\pi]$.

$$dx = -\sin t \, dt$$

e diventa un integrale di funzioni trigonometriche.

Altra possibilità:

$$\sqrt{1-x^2} = \sqrt{(1-x)(1+x)} = \sqrt{\frac{1-x}{1+x}(1+x)^2} = \sqrt{\frac{1-x}{1+x}}\,(1+x)$$

integrale già visto: si pone $t = \sqrt{\frac{1-x}{1+x}}$

Esempio:

$$\int \sqrt{2-4x^2}\, dx = (*)$$

1° modo

$$(*) = \sqrt{2} \int \sqrt{1-2x^2}\, dx =$$

$$\left[\begin{array}{l} \text{Poniamo } \sqrt{2}\,x = \cos t \qquad t \in [0,\pi] \\ \sqrt{1-2x^2} = \sqrt{1-\cos^2 t} = \sin t \\ dx = -\frac{1}{\sqrt{2}} \sin t \, dt \end{array}\right]$$

$$= \cancel{\sqrt{2}} \int \sin t \left(-\frac{1}{\cancel{\sqrt{2}}} \sin t\right) dt = -\int \sin^2 t \, dt$$

(integrale già visto: per parti oppure formule di bisezione)

e alla fine ci ricordiamo che $t = \arccos(\sqrt{2}\,x)$

2° modo

$$\sqrt{2} \int \sqrt{1-2x^2}\, dx =$$

$$\left[\begin{array}{l} \sqrt{2}x = t \\ dx = \dfrac{dt}{\sqrt{2}} \end{array}\right]$$

$$= \int \sqrt{1-t^2}\, dt$$

$$\sqrt{\frac{1-t}{1+t}} = s \implies 1-t = s^2(1+t)$$

$$t(1+s^2) = 1-s^2$$

$$t = \frac{1-s^2}{1+s^2}$$

$$dt = \frac{-2s(1+s^2) - (1-s^2)2s}{(1+s^2)^2}\, ds = \frac{-2s(1+\cancel{s^2}+1-\cancel{s^2})\,ds}{(1+s^2)^2}$$

$$= \frac{-4s}{(1+s^2)^2}\, ds$$

$$\sqrt{1-t^2} = \underbrace{\sqrt{\frac{1-t}{1+t}}}_{\shortparallel \atop s}\, \underbrace{(1+t)}\; = \frac{2s}{1+s^2}$$

$$1 + \frac{1-s^2}{1+s^2} = \frac{2}{1+s^2}$$

$$= \int \frac{2s}{1+s^2} \frac{(-4s)}{(1+s^2)^2}\, ds = -8 \int \frac{s^2}{(1+s^2)^3}\, ds$$

$$\frac{-8s^2}{(1+s^2)^3} = \frac{A}{1+s^2} + \frac{B}{(1+s^2)^2} + \frac{C}{(1+s^2)^3}$$

$$-8s^2 = \cancel{A(1+s^2)^2} + B(1+s^2) + C$$

grado 4: $\quad 0 = A$

grado 2: $\quad -8 = \cancel{2A} + B$

grado 0: $\quad 0 = B + C \quad \Rightarrow \quad C = 8$

E si conclude ricordando le formule per

$$\int \frac{ds}{1+s^2} = \operatorname{arctg} s \quad , \quad \int \frac{ds}{(1+s^2)^2} = \frac{1}{2}\left[\operatorname{arctg} s + \frac{s}{1+s^2}\right] + c$$

3° modo: <u>per parti</u>.

$$\int 1 \cdot \sqrt{2-4x^2}\,dx = x\sqrt{2-4x^2} + \int \frac{(4x^2 - 2) + 2}{\sqrt{2-4x^2}}\,dx =$$

per parti.

$$\left[\begin{array}{l} f'(x) = 1 \Rightarrow f(x) = x \\ g(x) = \sqrt{2-4x^2} \Rightarrow g'(x) = \dfrac{-8x}{2\sqrt{2-4x^2}} = \dfrac{-4x}{\sqrt{2-4x^2}} \end{array}\right]$$

$$= x\sqrt{2-4x^2} \boxed{- \int \sqrt{2-4x^2}\,dx} + \int \frac{2}{\sqrt{2-4x^2}}\,dx$$

$$\Rightarrow \int \sqrt{2-4x^2} = \frac{1}{2}\left[x\sqrt{2-4x^2} + \boxed{\int \frac{2}{\sqrt{2-4x^2}}\,dx}\right]$$

$$\| $$

$$\frac{2}{\sqrt{2}} \int \frac{dx}{\sqrt{1-2x^2}} = \sqrt{2} \int \frac{dx}{\sqrt{1-(\sqrt{2}x)^2}} =$$

$$= \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} \operatorname{arcsen}(\sqrt{2}x) + c$$

Controllare che nei tre modi suggeriti vengono funzioni che differiscono per una costante.

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5. $\int R\left(x, \sqrt{x^2+c}\right) dx$.

Varie sostituzioni possibili.

$\boxed{\sqrt{x^2+c} = x+t}$ cioè $t = \sqrt{x^2+c} - x$

$$\Downarrow$$

$$\cancel{x^2}+c = \cancel{x^2}+2xt+t^2$$

$$\boxed{x = \frac{c-t^2}{2t}} \qquad \boxed{dx = \frac{-2t^2-(c-t^2)}{2\ \ t^2}\,dt = -\frac{t^2+c}{2t^2}\,dt.}$$

$$\boxed{\sqrt{x^2+c} = x+t = \frac{c-t^2}{2t}+t = \frac{c+t^2}{2t}}$$

$$\int \frac{dx}{x+\sqrt{x^2+3}}$$

sost. $\boxed{\sqrt{x^2+3} = x+t}$

$$\cancel{x^2}+3 = \cancel{x^2}+2xt+t^2$$

$$\boxed{x = \frac{3-t^2}{2t}}$$

$$\boxed{\sqrt{x^2+3} = x+t = \frac{3-t^2}{2t}+t = \frac{3+t^2}{2t}}$$

$$\boxed{dx = \frac{-2t^2-3+t^2}{2\ \ t^2}\,dt = -\frac{3+t^2}{2t^2}\,dt}$$

L'integrale diventa

$$\int \frac{1}{\frac{3-t^2}{2t}+\frac{3+t^2}{2t}}\left(-\frac{3+t^2}{2t^2}\right)dt =$$

$$= -\int \frac{\cancel{2t}}{3-\cancel{t^2}+3+\cancel{t^2}} \frac{(3+t^2)}{\cancel{2}t^2}\, dt =$$

$$= -\frac{1}{6}\int \frac{3+t^2}{t}\, dt = -\frac{1}{6}\int \left(\frac{3}{t}+t\right) dt =$$

$$= -\frac{1}{6}\left(3 \log \cancel{|}t\cancel{|} + \frac{t^2}{2}\right)\Bigg|_{t=\sqrt{x^2+3}-x>0} + c.$$

togo il valore assoluto.

<u>Altro modo</u>: la presenza della quantità $\sqrt{3+x^2}$ suggerisce la seguente sostituzione

$$\boxed{x = \sqrt{3}\sinh t} \Rightarrow t = \text{settsinh}\, \frac{x}{\sqrt{3}}$$

$$\sqrt{3+x^2} = \sqrt{3+3\sinh^2 t} = \sqrt{3}\sqrt{1+\sinh^2 t} = \sqrt{3}\cosh t$$

$$dx = \sqrt{3}\cosh t\, dt.$$

$$\int \frac{1}{x+\sqrt{3+x^2}}\, dx = \int \frac{1}{\cancel{\sqrt{3}}\sinh t + \cancel{\sqrt{3}}\cosh t} (\cancel{\sqrt{3}}\cosh t)\, dt$$

$$= \int \frac{e^t+e^{-t}}{e^t-\cancel{e^{-t}}+e^t+\cancel{e^{-t}}}\, dt =$$

$$= \int \frac{e^t+e^{-t}}{2e^t}\, dt = \frac{1}{2}\int (1+e^{-2t})\, dt =$$

$$= \frac{1}{2}\left(t - \frac{1}{2}e^{-2t}\right) + c$$

$$= \frac{1}{2}\log(x+\sqrt{3+x^2}) - \frac{1}{4}\frac{1}{\left(\frac{x}{\sqrt{3}}+\sqrt{\frac{3+x^2}{3}}\right)^2}$$

$$\boxed{\begin{aligned} t &= \text{settsinh}\left(\frac{x}{\sqrt{3}}\right) = \\ &= \log\left(\frac{x}{\sqrt{3}}+\sqrt{1+\frac{x^2}{3}}\right) = \\ &= \log\frac{\left(x+\sqrt{3+x^2}\right)}{\sqrt{3}} = \end{aligned}}$$

N.B. Se compare

$\sqrt{2+4x^2}$, conviene porre

$$\sqrt{2+4x^2} = 2x + t$$

Calcoliamo l'integrale definito

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\, dx$$

Vari modi possibili:

1° modo

$$\sqrt{x^2-1} = x + t$$

$$\cancel{x^2} - 1 = \cancel{x^2} + 2xt + t^2$$

$$x = -\frac{1+t^2}{2t}$$

$$\sqrt{x^2-1} = x + t = \frac{-1-t^2+2t^2}{2t} = \frac{t^2-1}{2t}$$

$$dx = -\frac{1}{2}\,\frac{2t^2-(1+t^2)}{t^2}\,dt = \frac{1-t^2}{2t^2}\,dt$$

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{t^2-1}{\cancel{2t}}\left(-\frac{\cancel{2t}}{1+t^2}\right)\frac{1-t^2}{2t^2}\,dt =$$

$$= \frac{1}{2}\int \frac{(t^2-1)^2}{t^2(1+t^2)}\,dt$$

si procede con: 1) divisione
2) fratti semplici.

$$t = \sqrt{x^2-1} - x$$

2° modo

$\sqrt{x^2-1}$ suggerisce di porre:

$$\boxed{x = \cosh t}$$

$\boxed{\text{Attenzione: se fosse } \int_{-2}^{-1} \frac{\sqrt{x^2-1}}{x}\,dx \text{, dovremmo porre } x = -\cosh t \ (t>0)}$

$$\sqrt{x^2-1} = \sqrt{\cosh^2 t - 1} = |\sinh t| = \sinh t.$$

$$dx = \sinh t \, dt$$

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\sinh^2 t}{\cosh t}\,dt = \int R(e^t)\,dt \ldots$$

3° modo

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \frac{1}{2}\int 2x\,\frac{\sqrt{x^2-1}}{x^2}\,dx =$$

$x^2 = t$, $2x\,dx = dt$

$$= \frac{1}{2}\int \frac{\sqrt{t-1}}{t}\,dt =$$

$$\left\{\begin{array}{l} \sqrt{t-1} = s \\ t = s^2+1 \\ dt = 2s\,ds \end{array}\right.$$

$$= \frac{1}{\cancel{2}}\int \frac{s}{s^2+1}\,\cancel{2}s\,ds =$$

$$= \int \frac{s^2+1-1}{s^2+1}\,ds = \int \left(1 - \frac{1}{s^2+1}\right) ds =$$

$$= s - \operatorname{arctg} s + c \qquad \left(s = \sqrt{t-1} = \sqrt{x^2-1}\right)$$

$$= \sqrt{x^2-1} - \operatorname{arctg}\sqrt{x^2-1} + c.$$

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx = \left(\sqrt{x^2-1} - \operatorname{arctg}\sqrt{x^2-1}\right)\Big|_1^2 =$$

$$= \sqrt{3} - \operatorname{arctg}\sqrt{3} = \sqrt{3} - \frac{\pi}{3}$$

II

Attenzione: ci sono molte funzioni, anche molto semplici, le cui primitive non si possono scrivere in termini di funzioni elementari già note: per esempio

$$\int e^{x^2}dx\ ,\quad \int e^{-x^2}dx\ ,\quad \int \frac{\sin x}{x}\,dx\ ,\quad \int \sin(x^2)\,dx$$

$$\int \frac{e^x}{x}\,dx$$

Queste ammettono funzioni primitive,
per esempio, una primitiva di e^{x^2} è $F(x)=\int_0^x e^{t^2}dt$

ma essa non si può esprimere in termini di funz. elementari.

La funzione

$$\mathrm{Erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \qquad (\text{Error function})$$

è una funzione molto importante in probabilità e statistica.

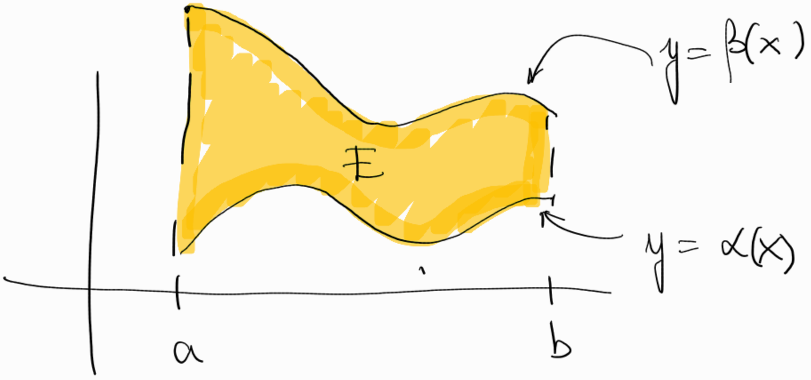
Calcolo delle aree

Considero insiemi del piano della seguente forma. (domini normali)

$$E=\{(x,y)\in\mathbb{R}^2:\ a\le x\le b,\ \ \alpha(x)\le y\le\beta(x)\}$$

dove $\alpha,\beta:[a,b]\to\mathbb{R}$ t.c. $\alpha(x)\le\beta(x)\ \forall x\in[a,b]$

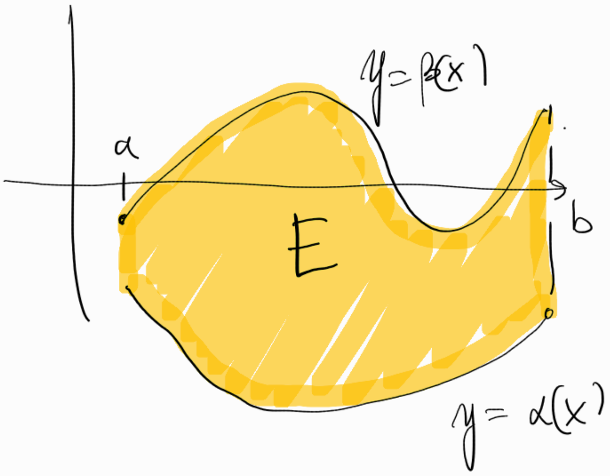
$\alpha,\beta\in\mathcal{R}([a,b])$.



In tal caso

$$\text{Area}(E) = \int_a^b \beta(x)\,dx - \int_a^b \alpha(x)\,dx = \int_a^b (\beta(x) - \alpha(x))\,dx$$

Questa formula è vera anche se α e β non sono funz. positive.



Per capirlo basta "spostare in alto" E.

cosa che non cambia l'area

$$\tilde{\beta}(x) = \beta(x) + c$$
$$\tilde{\alpha}(x) = \alpha(x) + c$$

$$\text{Area } E = \int_a^b [(\beta(x) + \cancel{c}) - (\alpha(x) + \cancel{c})]$$

Esempio:

Calcolare l'area di $E = \{(x,y):\ \underbrace{x-7}_{\alpha(x)} \leq y \leq \underbrace{2-3x^2-5x}_{\beta(x)}\}$

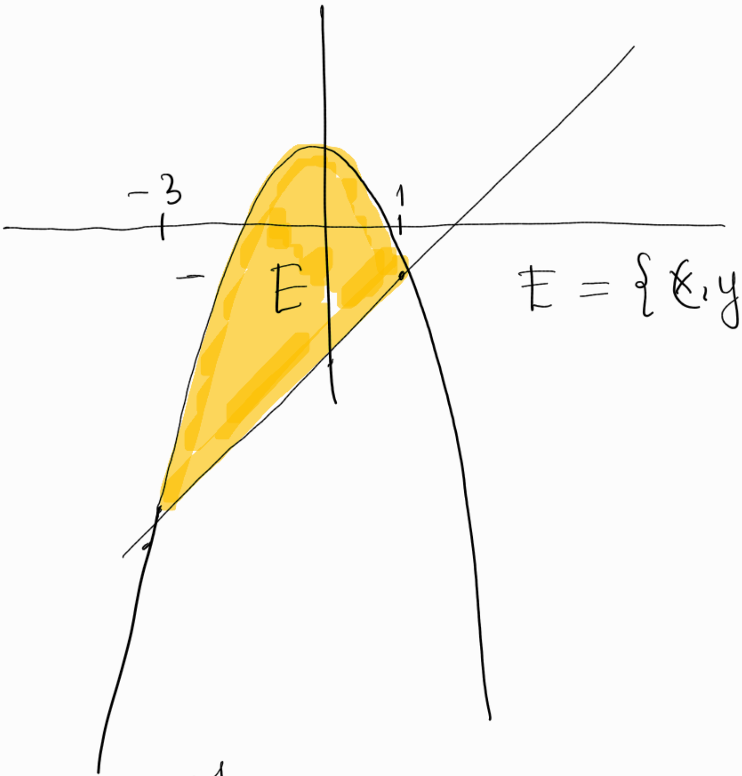
Calcolo le intersezioni tra retta $y = x - 7$

e parabola $y = 2 - 3x^2 - 5x$

$$x - 7 = 2 - 3x^2 - 5x$$

$$3x^2 + 6x - 9 = 0$$

$$x^2 + 2x - 3 = 0 \qquad x = -1 \pm \sqrt{1+3} = -1 \pm 2 = \begin{cases} -3 \\ 1 \end{cases}$$



$E = \{(x,y) : -3 \le x \le 1,\ x-7 \le y \le 2-3x^2-5x\}$

$$\text{Area } E = \int_{-3}^{1} \left[(2-3x^2-5x) - (x-7)\right] dx =$$

$$= \int_{-3}^{1} (-3x^2 - 6x + 9)\, dx$$

$$= \left(-x^3 - 3x^2 + 9x\right)\Big|_{-3}^{1} =$$

$$= -1 - 27 - 3(1-9) + 9(1+3) =$$

$$= -1 - 27 + 24 + 36 = 32$$

D: $\left(\sqrt{1+x^4} - 1\right)^2 \sim \left(\frac{x^4}{2}\right)^2 \sim \frac{x^8}{4}$

$\sqrt{1+t} - 1 \sim \frac{t}{2}$

N: $x^5 e^{x^3} - \log(1+x^5)$

$e^{x^3} = 1 + \underline{x^3} + o(x^3)$

$x^5\left(1 + x^3 + o(x^3)\right) - x^5 + o(x^8)$

$\log(1+t) = t - \frac{t^2}{2} + o(t^2)\ t \to 0$

$= \cancel{x^5} + x^8 + o(x^8) - \cancel{x^5} \sim x^8$

Calcolare l'area di

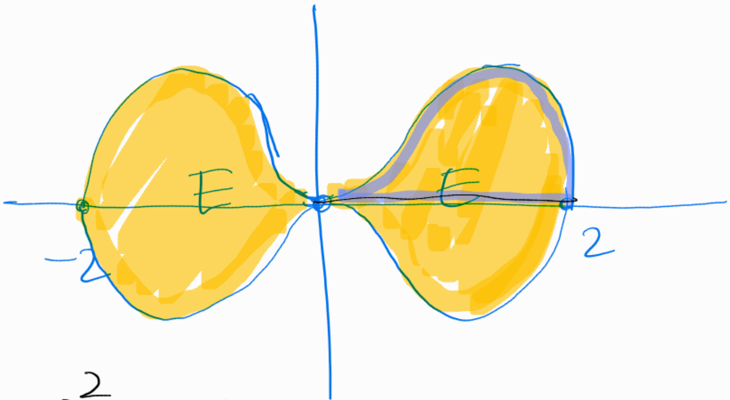
$$E = \{(x,y) : \quad y^4 \le \underbrace{x^{12}(16-x^4)}_{\geq 0}\}.$$

deve essere $\cancel{x^{12}}(16-x^4) \geq 0 \iff |x| \leq 2$

$$\boxed{-2 \leq x \leq 2}.$$

deve essere $|y| \leq |x|^3 \sqrt[4]{16-x^4}$.

cioè $-|x|^3\sqrt[4]{16-x^4} \leq y \leq \underbrace{|x|^3\sqrt[4]{16-x^4}}$



$$\text{Area } E = \int_{-2}^{2} \left(|x|^3\sqrt[4]{16-x^4} + |x|^3\sqrt[4]{16-x^4}\right) dx =$$

$$= 4\int_0^2 x^3\sqrt[4]{16-x^4}\, dx =$$

Pongo

$16-x^4 = t$

$-4x^3 dx = dt$

$x=0 \Rightarrow t=16$

$x=2 \Rightarrow t=0$

$$= \int_0^{16} \sqrt[4]{t}\, dt =$$

$$= \frac{4}{5} t^{5/4}\Big|_0^{16} = \frac{4}{5}\cdot 2^5 = \frac{128}{5}$$

Studio di funzioni integrali.

Studiare la funzione

$$h(x) = \int_0^x \cos(t^3)\,dt \quad \text{in} \quad \left[0, \sqrt[3]{\frac{3\pi}{2}}\right]$$

L'integrale non si calcola, ma so che

$h(0)=0$, $h'(x) = \cos(x^3)$.

$$h'(x) = 0 \iff \cos(x^3) = 0 \iff$$

$$\iff x^3 = (2k+1)\frac{\pi}{2}$$

$$\iff x = \sqrt[3]{(2k+1)\frac{\pi}{2}}$$

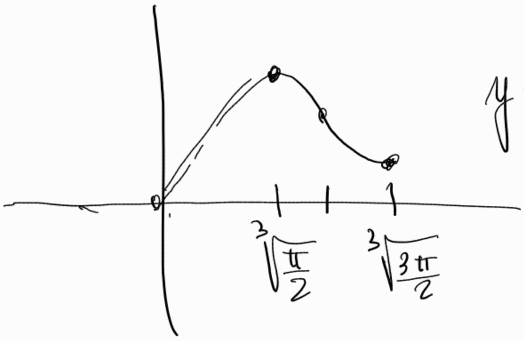
$$\iff x = \sqrt[3]{\frac{\pi}{2}}, \quad x = \sqrt[3]{\frac{3\pi}{2}}.$$

nell'intervallo scelto

$$h'(x) \geq 0 \iff \cos(x^3) > 0 \iff 0 \leq x^3 < \frac{\pi}{2}$$

$$\iff 0 \leq x \leq \sqrt[3]{\frac{\pi}{2}}$$

h strett. crescente in $\left[0, \sqrt[3]{\frac{\pi}{2}}\right]$, strett. decrescente in $\left[\sqrt[3]{\frac{\pi}{2}}, \sqrt[3]{\frac{3}{2}\pi}\right]$



$y = h(x)$ $\qquad$ $x = \sqrt[3]{\frac{\pi}{2}}$ pto di max assoluto.

$$h''(x) = -\sin(x^3)\,3x^2 \text{ etc...}$$

Calcolare la derivata di

$$g(x) = \int_{2x+1}^{\sin x} \sqrt[3]{2s^2+1}\, ds =$$

$$= \underbrace{\int_0^{\sin x} \sqrt[3]{2s^2+1}\, ds} - \int_0^{2x+1} \sqrt[3]{2s^2+1}\, ds.$$

$$h(t) = \int_0^t \sqrt[3]{2s^2+1}\, ds \Rightarrow h'(t) = \sqrt[3]{2t^2+1}$$

$$\int_0^{\sin x} \sqrt[3]{2s^2+1}\, ds = h(\sin x)$$

$$\left(\int_0^{\sin x} \sqrt[3]{2s^2+1}\, ds\right)' = (h(\sin x))' = h'(\sin x)\cdot \cos x =$$

$$= \sqrt[3]{2\sin^2 x+1}\cdot \cos x$$

$$\int_0^{2x+1} \sqrt[3]{2s^2+1}\, ds = h(2x+1)$$

$$\left(\int_0^{2x+1} \sqrt[3]{2s^2+1}\, ds\right)' = (h(2x+1))' = h'(2x+1)\, 2 =$$

$$= \sqrt[3]{2(2x+1)^2+1}\cdot 2$$

$$g'(x) = \sqrt[3]{2\sin^2 x+1}\cdot \cos x - 2\sqrt[3]{2(2x+1)^2+1}$$

$$\boxed{\frac{d}{dx}\left(\int_{\alpha(x)}^{\beta(x)} f(t)\, dt\right) = f(\beta(x))\,\beta'(x) - f(\alpha(x))\,\alpha'(x)}$$

se f continua in J $\quad \alpha(x), \beta(x) : I \longrightarrow J$ derivabili.