Abbiamo visto:

$$\boxed{\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \qquad \forall \alpha \in \mathbb{R}}$$

Questo si può scrivere anche così:

$$(1+x)^\alpha - 1 = \alpha x \, (1 + o(1)) \qquad \text{per } x \to 0$$

$$(1+x)^\alpha - 1 \sim \alpha x \qquad \text{"}$$

$$(1+x)^\alpha - 1 = \alpha x + o(x) \qquad \text{"}$$

$$(1+x)^\alpha = 1 + \alpha x + o(x) \qquad \text{"}$$

$$\cancel{(1+x)^\alpha \sim 1 + \alpha x + o(x)}$$

Questo vuol dire solo: "$(1+x)^\alpha \to 1$ per $x \to 0$"

Per es.: $\underbrace{\sqrt{1+x}}_{=(1+x)^{1/2}} = 1 + \frac{1}{2}x + o(x)$ per $x \to 0$

$(1+x)^2 = 1 + 2x + \underbrace{o(x)}_{=x^2}$ per $x \to 0$

$$\lim_{x\to+\infty} \left(\underbrace{\sqrt[3]{x^3+2x^2}}_{\sim x} - x\right)$$

1° modo: (già noto) si usa la formula $A^3 - B^3 = (A-B)(A^2+AB+B^2)$

$$\frac{\left(\sqrt[3]{x^3+2x^2} - x\right)\cdot\left((x^3+2x^2)^{2/3} + x\sqrt[3]{x^3+2x^2} + x^2\right)}{\left((x^3+2x^2)^{2/3} + x\sqrt[3]{x^3+2x^2} + x^2\right)} =$$

$$= \frac{\cancel{x^3}+2x^2-\cancel{x^3}}{\left((x^3+2x^2)^{2/3} + x\sqrt[3]{x^3+2x^2} + x^2\right)} = \frac{2\cancel{x^2}}{\cancel{x^2}\underbrace{\left(\left(1+\frac{2}{x}\right)^{2/3} + \sqrt[3]{1+\frac{2}{x}} + 1\right)}_{\to 3}}$$

$\to \frac{2}{3}$.

2° modo (nuovo!)

$$\sqrt[3]{x^3+2x^2} - x = x\overbrace{\left(\sqrt[3]{1+\frac{2}{x}} - 1\right)}^{\sim \frac{2}{3x}} \sim x\cdot\frac{2}{3x} = \frac{2}{3}$$

$\frac{2}{x} \to 0$

$\sqrt[3]{1+t} - 1 \sim \frac{t}{3}$ $t \to 0$

$\Rightarrow \sqrt[3]{1+\frac{2}{x}} - 1 \sim \frac{2}{3x}$ $x \to +\infty$

oppure $\cancel{x}\boxed{\dfrac{\left(\sqrt[3]{1+\frac{2}{x}}-1\right)}{\frac{2}{x}}}\,\dfrac{2}{\cancel{x}} \to \dfrac{2}{3}$

$$\frac{(1+t)^{1/3}-1}{t} \searrow \frac{1}{3}$$

con $t=\frac{2}{x}\to 0$

$$\lim_{x\to+\infty}\left(\sqrt[8]{x^8+2x^7}-x\right)=(+\infty-\infty)$$

1° modo:

$$A^8-B^8=(A-B)\left(A^7+A^6B+A^5B^2+\dots+B^7\right)$$

verifica immediata.

Provare a farlo.

2° modo:

$$\sqrt[8]{x^8+2x^7}-x = x\boxed{\left(\sqrt[8]{1+\frac{2}{x}}-1\right)} \sim \frac{x}{4x}=\frac{1}{4}$$

$\sim \frac{1}{4x}$ per $x\to+\infty$

OSS $\sqrt[8]{1+t}-1=(1+t)^{1/8}-1\sim\frac{t}{8}$ $\quad t\to 0$

$$\sqrt[8]{1+\frac{2}{x}}-1 \sim \frac{2}{8x}=\frac{1}{4x}$$

$\frac{2}{x}=t\to 0$

$x\to+\infty$

Determinare $c\in\mathbb{R}$ t.c. $\displaystyle\lim_{x\to-\infty}\left(\frac{x+c}{x-c}\right)^x=4.$

$$\lim_{x\to-\infty}\left(\frac{x+c}{x-c}\right)^x=(1^{-\infty})$$

$$\left(\frac{x+c}{x-c}\right)^x = \left[\underbrace{\left(1+\frac{2c}{x-c}\right)^{x-c}}_{\downarrow\, e^{2c}}\right]\underbrace{\left(1+\frac{2c}{x-c}\right)^{c}}_{\downarrow\, 1} \to e^{2c} \qquad \boxed{x\to-\infty}$$

Devo risolvere $e^{2c}=4 \iff 2c=\log 4 = 2\log 2$

$$\boxed{c=\log 2} \leftarrow$$

Oppure così

$$\left(\frac{x+c}{x-c}\right)^x = e^{x\log\left(\frac{x+c}{x-c}\right)} \longrightarrow e^{2c}$$

$$\underset{\substack{\downarrow\\-\infty}}{x}\ \underbrace{\log\left(\frac{x+c}{x-c}\right)}_{\substack{\downarrow\\0}} = x\,\boxed{\log\left(1+\frac{2c}{x-c}\right)} \sim \frac{2cx}{x-c} \xrightarrow[x\to-\infty]{} 2c$$

$$\sim \frac{2c}{x-c}$$

$\log(1+t)\sim t \qquad t\to 0$

$$\lim_{t\to 0}\frac{\log(1+t)}{t}=1$$

$\Rightarrow \log\left(1+\frac{2c}{x-c}\right)\sim\frac{2c}{x-c}$ per $x\to-\infty$ $\qquad t=\frac{2c}{x-c}\to 0$ quando $x\to-\infty$

$$\lim_{x\to 0}\left(\underbrace{\log\left(3^{-1/x}\right)}_{=\,-\frac{1}{x}\log 3} - \frac{1}{x}\log\left[\left(x+\frac{1}{3}\right)(1-2x)\right]\right)$$

$$= \lim_{x\to 0}\left[-\frac{1}{x}\left(\log 3+\log\left[\left(x+\frac{1}{3}\right)(1-2x)\right]\right)\right]$$

$$-\lim_{x\to0}\left[-\frac{1}{x}\underbrace{\log\left[(3x+1)(1-2x)\right]}_{\to 0}\right]$$

$$=-\lim_{x\to0}\frac{1}{x}\cdot\boxed{\frac{\log(1+\overbrace{x-6x^2}^{\to0})}{x-6x^2}}(x-6x^2)=-1$$

$$\boxed{\ }\to 1$$

$$\lim_{t\to0}\frac{\log(1+t)}{t}=1$$

$$\lim_{x\to4^+}\frac{\log(5e^{x-4}-x)-\sin(3x-12)}{x-4}=\left(\frac{0}{0}\right)$$

$x-4\to0^+$

$$x-4=y\to0^+$$
$$x=4+y$$

$$=\lim_{y\to0^+}\frac{\log(5e^y-4-y)-\sin(3y)}{y}=$$

$$=\lim_{y\to0^+}\left[\frac{\log(5e^y-4-y)}{y}-\boxed{\frac{\sin(3y)}{y}}\right]=4-3=1$$

$$\boxed{\ }\to 3$$

$y\to0^+$

$$\frac{\log(5e^y-4-y)}{y}=\frac{\log(1+(5e^y-5-y))}{y}\sim \quad \sim 5e^y-5-y$$

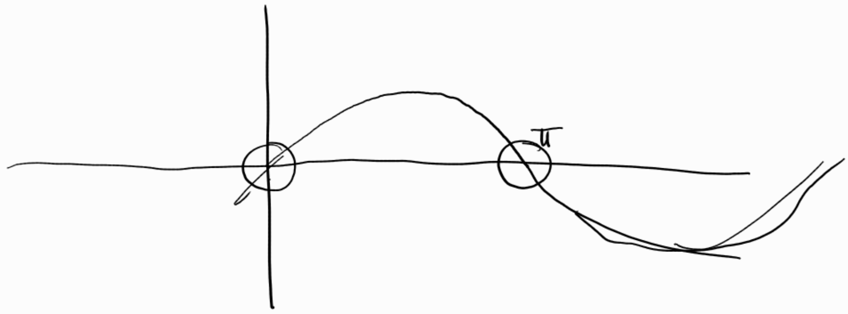
$$\sim\frac{5e^y-5-y}{y}=5\boxed{\frac{(e^y-1)}{y}}-1\Rightarrow 5-1=4$$

$$\boxed{\ }\to 1$$

$$\lim_{x\to\pi}\frac{\sin x}{x-\pi} = \left(\frac{0}{0}\right) =$$

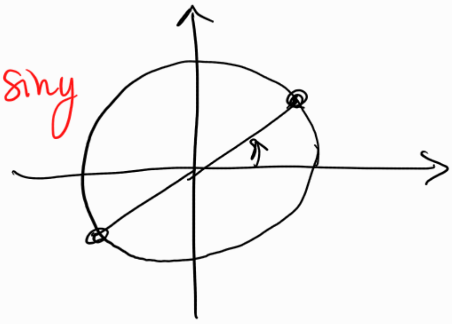
$y = x-\pi \to 0$

$x = y+\pi.$



$$= \lim_{y\to 0}\frac{\sin(y+\pi)}{y} =$$

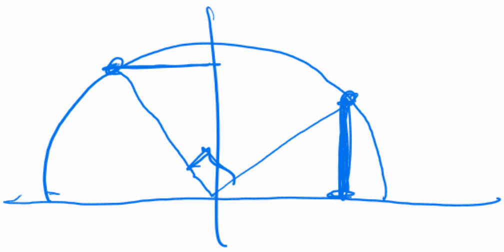
OSS $\sin(y+\pi) = -\sin y$



$$= \lim_{y\to 0}\frac{-\sin y}{y} = -1.$$

$$\lim_{n\to+\infty} \underbrace{(2n+5)}_{\downarrow\,+\infty} \underbrace{\boxed{\cos\left(\frac{\pi(n^2+1)}{2n^2+n}\right)}}_{\downarrow\,\cos\frac{\pi}{2}=0} = (+\infty\cdot 0)$$

$$\underbrace{(2n+5)}_{\wr\, 2n}\ \underbrace{\cos\left(\frac{\pi(n^2+1)}{2n^2+n}\right)}_{\cos\left(\frac{\pi}{2}+\underbrace{\left(\frac{\pi(n^2+1)}{2n^2+n}-\frac{\pi}{2}\right)}_{\downarrow\,0}\right)} \sim -2n\sin\left(\underbrace{\frac{\pi(n^2+1)}{2n^2+n}-\frac{\pi}{2}}_{\downarrow\,0}\right)$$



[OSS $\cos\left(\frac{\pi}{2}+t\right) = -\sin t$]

$$\underbrace{(2n+5)}\ \cos\left(\frac{\pi(n^2+1)}{2n^2+n}\right) \sim -2n\sin\left(\underbrace{\frac{\pi(n^2+1)}{2n^2+n}-\frac{\pi}{2}}_{\downarrow\,0}\right) \sim$$

$$\sim -2n\left(\frac{\pi(n^2+1)}{2n^2+n} - \frac{\pi}{2}\right) = -2\pi n\left(\frac{n^2+1}{2n^2+n} - \frac{1}{2}\right) =$$

$$= -\cancel{2}\pi n \, \frac{\cancel{2n^2}+2-\cancel{2n^2}-n}{\cancel{2}(2n^2+n)} = \pi \, \frac{n^2-2n}{2n^2+n} =$$

$$= \pi \, \frac{n-2}{2n+1} \longrightarrow \frac{\pi}{2}$$

$$\lim_{x\to 0} \frac{e^{tg^3 x}-1}{x(\cos x - e^{x^2})} = \left(\frac{0}{0}\right) = -\frac{2}{3}$$

$$e^{tg^3 x} - 1 \sim tg^3 x \sim x^3 \qquad x\to 0$$

$e^t - 1 \sim t$ per $t \to 0$.

$t = tg^3 x \to 0$ per $x \to 0$.

$\boxed{x \to 0}$

$tg\, x \sim x$

$\Rightarrow (tg\, x)^3 \sim x^3$

$$\cos x - e^{x^2} = \left(\underbrace{\boxed{\frac{\cos x - 1}{x^2}}}_{-1/2} + \underbrace{\boxed{\frac{1-e^{x^2}}{x^2}}}_{-1}\right) x^2 \sim -\frac{3}{2}x^2$$

$$\frac{e^{tg^3 x}-1}{x(\cos x - e^{x^2})} \sim \frac{\cancel{x^3}}{\cancel{x}\cdot\left(-\frac{3}{2}\cancel{x^2}\right)} = -\frac{2}{3}$$

$$\frac{e^{tg^3 x}-1}{tg^3 x}\;\frac{tg^3 x}{x^3}\;\frac{\cancel{x^3}\,x^2}{\cancel{x}\,(\cos x - e^{x^2})} = \frac{(1+o(1))\,1}{\underbrace{\left(\frac{\cos x - e^{x^2}}{x^2}\right)}_{\to -\frac{3}{2}}} =$$

$$\frac{e^{tg^3 x}-1}{tg^3 x} \to 1 \qquad \frac{tg^3 x}{x^3} \to 1$$

$$\frac{\cos x - e^{x^2}}{x^2} = \underbrace{\frac{\cos x - 1}{x^2}}_{\to -1/2} - \underbrace{\frac{e^{x^2}-1}{x^2}}_{\to 1}$$

$$\lim_{x\to 0} \boxed{\frac{1-\cos x}{x^2}}\, x = 0 \qquad \frac{1-\cos x}{x^2} \to 1/2 \qquad x \to 0$$

Limiti di funzioni monotòne

Ricordo che: per una succ^{ne} crescente (decrescente) $\{a_n\}$

$$\exists \lim_{n\to+\infty} a_n = \sup_{n\in\mathbb{N}} a_n \quad (\inf)$$

Con le dovute modifiche, vale un teorema simile per le funzioni:

TEOREMA Sia $f: X \subseteq \mathbb{R} \to \mathbb{R}$ crescente (decrescente)

(cioè: se $x_1, x_2 \in X$, $x_1 < x_2$ $\Rightarrow f(x_1) \le f(x_2)$)

Allora:

1) Sia $x_0 \in \mathbb{R}^*$ un pto di accum. sinistro di X.

(Cioè: ogni intorno sinistro di x_0: $(x_0-\delta, x_0)$ se $x_0 \in \mathbb{R}$
$(M, +\infty)$ se $x_0 = +\infty$
contiene infiniti punti di X)

$$\Rightarrow \exists \lim_{x\to x_0^-} f(x) = \boxed{\sup_{x\in X\cap(-\infty, x_0)} f(x)} \quad (\inf)$$

2) Sia $x_0 \in \mathbb{R}^*$ un pto di accum^{ne} destro per X.

$$\Rightarrow \quad \exists \lim_{x \to x_0^+} f(x) = \inf_{x \in X \cap (x_0, +\infty)} f(x) \quad \text{sup}$$