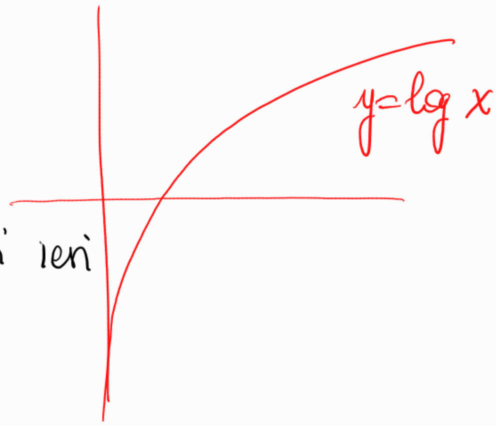
$$\lim_{x\to 0^+} x^x = (0^0) = 1$$

$$x^x = e^{x\log x} \xrightarrow[\text{per}]{} e^0$$



$$\underbrace{x}_{\downarrow\, 0}\,\underbrace{\log x}_{\downarrow\, -\infty} = 0\cdot(-\infty) \to 0$$

per uno dei limiti visti ieri

$$\lim_{x\to 0^+} x^\alpha \log_b x = 0 \qquad \forall \alpha>0, \quad \forall b>0,\ b\neq 1$$

$$\lim_{n\to+\infty} \sqrt[n]{n} = \lim_{n\to+\infty} n^{1/n} = (+\infty^0) = 1$$

$$n^{1/n} = e^{\frac{1}{n}\log n} \to e^0 = 1$$

$$\frac{\log n}{n} \longrightarrow 0$$ (perché il log "perde" con le potenze)

$$\lim_{n\to+\infty} \sqrt[n]{3n^4-2n^2+6} = \lim_{n\to+\infty} (3n^4-2n^2+6)^{1/n} = (+\infty^0) = 1$$

$$(3n^4-2n^2+6)^{1/n} = e^{\frac{1}{n}\log(3n^4-2n^2+6)} \to e^0 = 1$$

$$\left(\frac{+\infty}{+\infty}\right)$$

$$\frac{1}{n}\log(3n^4-2n^2+6) = \frac{\log(n^4(3+o(1)))}{n} =$$

$$= \frac{4\log n + \log(3+o(1))}{n} = \underbrace{4\frac{\log n}{n}}_{\downarrow\, 0} + \underbrace{\frac{\log(3+o(1))}{n}}_{\downarrow\, 0} \searrow 0$$

Allo stesso modo si dimostra che

$$\lim_{n\to+\infty} \sqrt[n]{\underbrace{P_k(n)}_{\text{polinomio in } n}}$$

$$\lim_{x\to+\infty} \frac{x}{e^{\sqrt{\log x}}} = \left(\frac{+\infty}{+\infty}\right) = +\infty.$$

$$\frac{x}{e^{\sqrt{\log x}}} = \frac{e^{\log x}}{e^{\sqrt{\log x}}} = e^{\log x - \sqrt{\log x}} \Longrightarrow e^{+\infty} = +\infty$$

$$\log x - \sqrt{\log x} = \underbrace{\log x}_{\downarrow \, +\infty} \underbrace{\left(1 - \frac{1}{\sqrt{\log x}}\right)}_{\downarrow \, 1} \longrightarrow +\infty$$

Limiti notevoli di funzioni trigonometriche

1) $$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

Si scrive anche così:

$$\boxed{\sin x = x\,(1 + o(1)) \qquad \text{per } x \to 0}$$

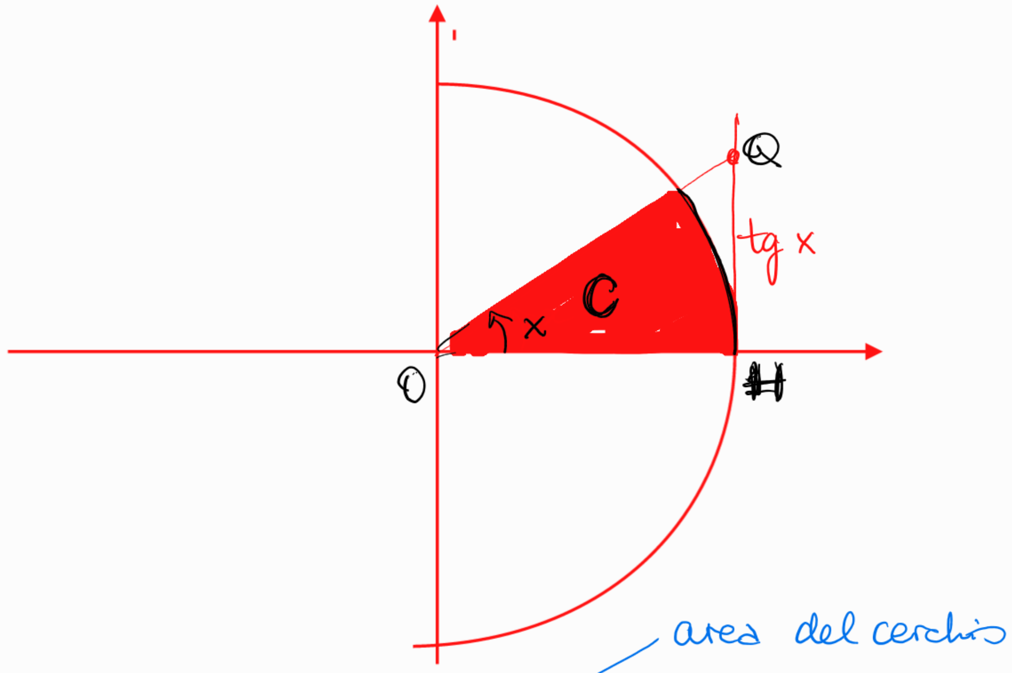
$$\boxed{\sin x \sim x \qquad \text{per } x \to 0}$$

$$\boxed{\sin x = x + o(x)} \qquad \text{per } x \to 0.$$

DIM

Lemma $\forall x \in \left[0, \frac{\pi}{2}\right)$ $\boxed{\sin x \leq x \leq \operatorname{tg} x}$

già dim.



$$\frac{\text{Area } C}{x} = \frac{\pi}{2\pi} \Rightarrow \text{Area } C = \frac{x}{2}$$

$$\frac{x}{2} = \text{Area } C \le \text{Area } O\hat{H}Q = \frac{1 \cdot \text{tg } x}{2}$$

$$\frac{x}{2} \le \frac{\text{tg } x}{2} \iff x \le \text{tg } x. \qquad \square$$

se $0 < x < \frac{\pi}{2}$ $\qquad \sin x \le x \le \text{tg } x = \dfrac{\sin x}{\cos x}$

divido per $\sin x > 0$ $\qquad 1 \le \dfrac{x}{\sin x} \le \dfrac{1}{\cos x}$

Passo ai reciproci $\qquad \cos x \le \dfrac{\sin x}{x} \le 1$

Facciamo $x \to 0^+$
e usiamo il teor. dei carab. $\qquad \cos 0 = 1 \qquad 1$

Abbiamo provato che $\displaystyle\lim_{x \to 0^+} \frac{\sin x}{x} = 1$.

OSS $\quad \dfrac{\sin x}{x}$ è una f. pari.

$$\Rightarrow \lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

$\square$

2) $$\boxed{\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}}$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{\overbrace{(1-\cos x)(1+\cos x)}^{1-\cos^2 x = \sin^2 x}}{x^2 (1+\cos x)} =$$

$$= \lim_{x\to 0} \underbrace{\boxed{\left(\frac{\sin x}{x}\right)^2}}_{\downarrow\, 1} \underbrace{\frac{1}{1+\cos x}}_{\downarrow\, 1/2} = \frac{1}{2}.$$

Il limite si scrive anche come

$$1-\cos x = \frac{x^2}{2}\left(1+o(1)\right) \qquad \text{per } x\to 0$$

$$1-\cos x = \frac{x^2}{2} + \underbrace{\frac{x^2}{2} o(1)}_{= o(x^2)} \qquad ''$$

$$\boxed{\cos x = 1 - \frac{x^2}{2} + o(x^2)} \qquad ''$$

$$\cancel{\cos x \sim 1 - \frac{x^2}{2}}$$ ← è vero ma non ha lo stesso significato. Questo vuol solo dire che $\cos x \to 1$ per $x\to 0$

3) $$\boxed{\lim_{x\to 0} \frac{\operatorname{tg} x}{x} = 1}$$

$$\frac{\operatorname{tg} x}{x} = \frac{\sin x}{x \cos x} \to 1$$

$\frac{\sin x}{x} \to 1$, $\cos x \to 1$

Il limite si scrive anche

$$\left.\begin{aligned} \operatorname{tg} x &= x\,(1 + o(1)) \\ \operatorname{tg} x &= x + o(x) \\ \operatorname{tg} x &\sim x \end{aligned}\right\} \text{ per } x \to 0.$$

$$\lim_{x\to 0} \frac{x \sin x}{\cos x - 1} = \left(\frac{0}{0}\right) = -2$$

$$\frac{x \sin x}{\cos x - 1} = \underbrace{\frac{x^2}{\cos x - 1}}_{-2} \; \underbrace{\frac{\sin x}{x}}_{1} \to -2$$

moltiplico e divido per x

Più sinteticamente:

$$\frac{x \overbrace{\sin x}^{\sim x}}{\underbrace{\cos x - 1}_{\sim -\frac{x^2}{2}}} \sim \frac{\cancel{x^2}}{\left(-\frac{\cancel{x^2}}{2}\right)} = -2$$

posso sostituire perché sono prodotti e/o frazioni.

$$\lim_{x\to 0} \frac{\operatorname{tg} x - \sin^2 x}{x} = \left(\frac{0}{0}\right) = 1.$$

$$\frac{\operatorname{tg} x - \sin^2 x}{x} = \underbrace{\frac{\operatorname{tg} x}{x}}_{\to 1} - \frac{\sin^2 x}{x} \sim \frac{x^2}{x} = x \to 0 \quad \to 1$$

$$\frac{\overbrace{\operatorname{tg} x}^{\sim x} - \overbrace{\sin^2 x}^{\sim x^2}}{x} \approx \cancel{\frac{x - x^2}{x}}$$

Attenzione non si può sost. un asint. equivalente in una somma.

$$\overbrace{\operatorname{tg} x}^{\sim x} - \overbrace{\sin^2 x}^{\sim x^2} = \operatorname{tg} x \left(1 - \boxed{\frac{\sin^2 x \ {}^{\sim x^2}}{\operatorname{tg} x \ {}_{\sim x}}}\right) \sim \operatorname{tg} x \sim x$$

$$\boxed{\dfrac{\sin^2 x}{\operatorname{tg} x}} \to 0$$

$$\lim_{x\to 0} \frac{2 - 2\cos x}{\operatorname{sen}(x^2) - 3\sin x - x^2} = \left(\frac{0}{0}\right) = 0.$$

$$2 - 2\cos x = 2(1-\cos x) \sim x^2$$

$$\frac{2 - 2\cos x}{\underbrace{\operatorname{sen}(x^2)}_{\sim x^2} - \underbrace{3\sin x}_{\sim 3x} - x^2} \approx \frac{x^2}{x\left(\underbrace{\frac{\sin(x^2)}{x}}_{\to 0} - \underbrace{3\frac{\sin x}{x}}_{\to -3} - \underbrace{\frac{x^2}{x}}_{\to 0}\right)} \sim \frac{x^{\cancel{2}}}{-3\cancel{x}} = -\frac{x}{3}$$

Limiti notevoli che coinvolgono e.

1) $$\boxed{\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e}$$

E' vero per il teor. ponte, infatti abbiamo già provato che se $a_n \to \pm\infty$ allora $\left(1+\frac{1}{a_n}\right)^{a_n} \to e$.

$$\lim_{x\to\pm\infty}\left(1-\frac{1}{x}\right)^x =$$

Poniamo $y=-x \to -\infty$

$$= \lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^{-y} = \lim_{y\to-\infty}\frac{1}{\left(1+\frac{1}{y}\right)^y} = \frac{1}{e}.$$

2) $$\boxed{\lim_{x\to\pm\infty}\left(1+\frac{a}{x}\right)^x = e^a}$$

Se $a=0$, la funzione vale $1^x = 1 \Rightarrow$ il limite vale 1.

Se $a\neq 0$ $$\left(1+\frac{a}{x}\right)^x = \Bigg[\underbrace{\left(1+\frac{1}{(x/a)}\right)^{x/a}}_{\to e}\Bigg]^a \to e^a$$

oss $\frac{x}{a} \to \pm\infty$

$\frac{x}{a} = y$

$$\lim_{x\to+\infty}\left(1+\frac{1}{x+1}\right)^{x^2} = \left(1^{+\infty}\right) = +\infty.$$

$$\left(1+\frac{1}{x+1}\right)^{x^2} = \left[\underbrace{\left(1+\frac{1}{x+1}\right)^{x+1}}_{\downarrow\, e}\right]^{\overset{\frac{x^2}{x+1}}{\searrow +\infty}} \to e^{+\infty} = +\infty$$

$$\lim_{x\to 0} (1+x)^{1/x} = (1^{\pm\infty})$$

$$\lim_{x\to 0^+} (1+x)^{1/x} = (1^{+\infty})$$

$$\left[y = \frac{1}{x} \to +\infty\right] \qquad = \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y} = e$$

$$\lim_{x\to 0^-} (1+x)^{1/x} = \qquad \left[y = \frac{1}{x} \to -\infty\right]$$

$$= \lim_{y\to -\infty}\left(1+\frac{1}{y}\right)^{y} = e$$

Quindi:

3) $$\boxed{\lim_{x\to 0} (1+x)^{1/x} = e}$$

4) $$\lim_{x\to 0} \frac{1}{x}\log(1+x) = \lim_{x\to 0} \log\left((1+x)^{1/x}\right) = \log e = 1$$

$$\boxed{\lim_{x\to 0} \frac{1}{x}\log(1+x) = 1}$$

Si può scrivere anche:

$$\log(1+x) \sim x \qquad \text{per } x\to 0$$

$$\log(1+x) = x(1+o(1)) \qquad \text{per } x \to 0$$

$$\log(1+x) = x + o(x) \qquad \text{per } x \to 0$$

$$\log t \sim t-1 \qquad \boxed{t \to 1}$$

Attenzione: questo è vero solo se la base del log è e.

$$\boxed{\lim_{x\to 0} \frac{\log_b(1+x)}{x} = \log_b e = \frac{1}{\log b}}$$

5) $$\boxed{\lim_{x\to 0} \frac{e^x - 1}{x} = 1}$$

$$\lim_{x\to 0} \frac{e^x-1}{x} = \left(\frac{0}{0}\right) =$$

$$e^x - 1 = y \to 0$$

$$e^x = 1 + y$$

$$x = \log(1+y)$$

$$= \lim_{y\to 0} \frac{y}{\log(1+y)} = 1.$$

Si serve anche: $e^x - 1 \sim x$ per $x \to 0$

$$e^x - 1 = x(1+o(1)) \qquad ''$$

$$e^x - 1 = x + o(x) \qquad ''$$

$$e^x = 1 + x + o(x) \qquad ''$$

$$\lim_{x\to 0} \frac{2^x - 1}{x} = \lim_{x\to 0} \underbrace{\boxed{\frac{e^{x\log 2} - 1}{x \log 2}}}_{1} \log 2 = \log 2.$$

5')
$$\boxed{\lim_{x\to 0} \frac{a^x-1}{x} = \log a. \qquad \forall a>0.}$$

6)
$$\boxed{\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \qquad \forall \alpha\in\mathbb{R}.}$$

$$\lim_{x\to 0} \frac{(1+x)^\alpha-1}{x} = \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{e^{\boxed{\alpha\log(1+x)}\to 0}-1}{x} =$$

$$= \lim_{x\to 0} \boxed{\frac{e^{\alpha\log(1+x)}-1}{\alpha\log(1+x)}}\ \alpha\ \boxed{\frac{\log(1+x)}{x}} = \alpha$$

$$\frac{e^{\alpha\log(1+x)}-1}{\alpha\log(1+x)} = \frac{e^t-1}{t} \to 1 \quad \text{con } t\to 0$$

$$\frac{\log(1+x)}{x} \to 1$$