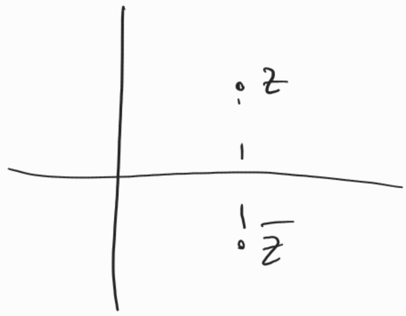
Equazioni nei numeri complessi

$$z + i\left(\bar{z}\right)^2 = -2i$$

$\boxed{x, y \in \mathbb{R}.}$

$z = x + iy$

$\bar{z} = x - iy$



$(\bar{z})^2 = (x - iy)^2 = x^2 - y^2 - 2ixy$

$i(\bar{z})^2 = ix^2 - iy^2 + 2xy$

L'eq^{ne} diventa

$$\boxed{x + iy + ix^2 - iy^2 + 2xy = -2i}$$

Questo corrisponde a un sistema di due eq^{ne} per parte reale e parte immaginaria.

$$\begin{cases} x + 2xy = 0 \\ y + x^2 - y^2 = -2 \end{cases} \qquad \begin{cases} x(1 + 2y) = 0 \\ \text{——} \quad \text{————} \end{cases}$$

$$\begin{cases} x = 0 \\ y^2 - y - 2 = 0 \end{cases} \quad \vee \quad \cancel{\begin{cases} y = -\frac{1}{2} \\ x^2 = -2 + \frac{1}{4} + \frac{1}{2} < 0 \end{cases}}$$

$(y = 2) \vee (y = -1)$ non ha soluzioni

Soluzioni sono $\boxed{z = 2i}$ $\boxed{z = -i}$

$$z+\bar{z}-3\,\mathrm{Im}(z)=z^2+|z|$$

$z=x+iy \qquad x,y\in\mathbb{R}$

$\bar{z}=x-iy$

$\mathrm{Im}(z)=y$

$z^2=(x+iy)^2=x^2-y^2+2ixy \qquad |z|=\sqrt{x^2+y^2}$

$$\boxed{x+\cancel{iy}+x-\cancel{iy}-3y=x^2-y^2+2ixy+\sqrt{x^2+y^2}}$$

Il sistema corrispondente è

$$\begin{cases} 2x-3y=x^2-y^2+\sqrt{x^2+y^2} \\ \cancel{2}xy=0 \end{cases}$$

$$\begin{cases} x=0 \\ -3y=-y^2+|y| \end{cases} \quad \vee \quad \begin{cases} y=0 \\ 2x=x^2+|x| \end{cases}$$

$\boxed{x=0}$

$y^2-3y-|y|=0$

$$\begin{cases} y\geq 0 \\ y^2-4y=0 \end{cases} \qquad \boxed{y=0 \quad y=4}$$

$$\begin{cases} y<0 \\ y^2-2y=0 \end{cases} \qquad \cancel{y=0} \quad \cancel{y=2}$$

Sol^{ni} del 1° sistema $\boxed{z=0}$ $\boxed{z=4i}$

Passiamo al 2° sistema

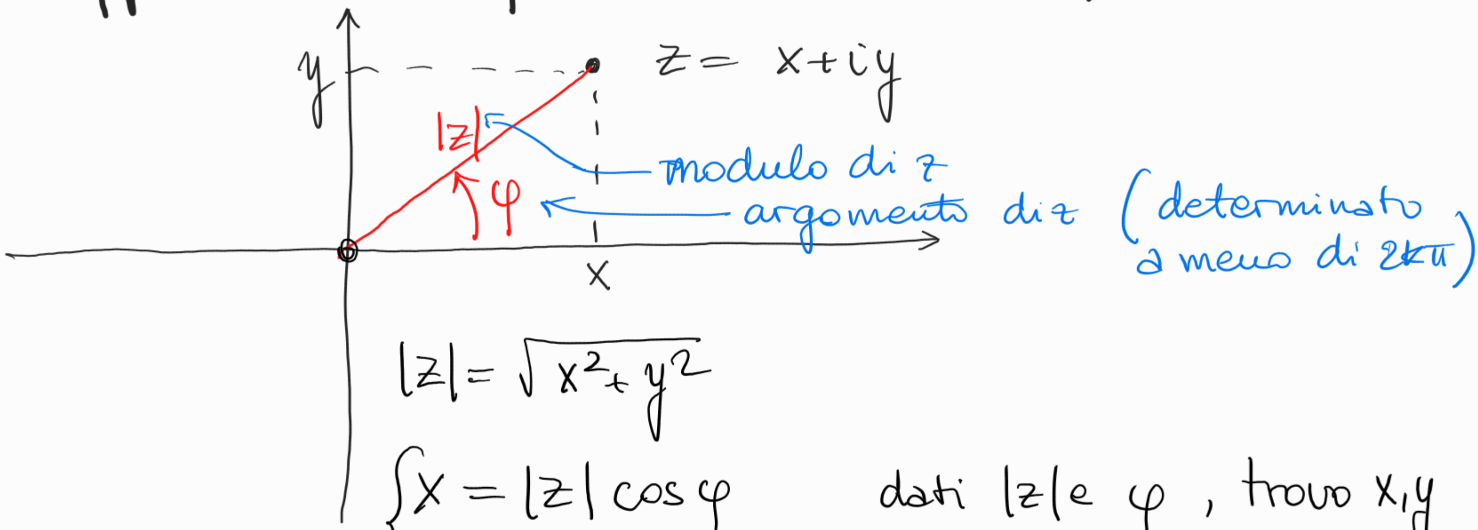
$$\begin{cases} y=0 \\ x^2-2x+|x|=0 \end{cases}$$

$$\begin{cases} x\geq 0 \\ x^2-x=0 \end{cases} \qquad x=0 \text{ già trovata} \quad x=1$$

$$\begin{cases} x<0 \\ x^2-3x=0 \end{cases} \qquad \cancel{x=0} \quad \cancel{x=3}$$

Ulteriore soluzione $\boxed{z=1}$

Rappresentazione polare dei numeri complessi



$$|z| = \sqrt{x^2+y^2}$$

$$\begin{cases} x = |z| \cos\varphi \\ y = |z| \sin\varphi \end{cases}$$ dati $|z|$ e φ, trovo x, y

Dati x, y, come trovo $|z|$ e φ?

$|z| = \sqrt{x^2+y^2}$, $\boxed{\begin{cases} \cos\varphi = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{|z|} \\ \sin\varphi = \dfrac{y}{\sqrt{x^2+y^2}} = \dfrac{y}{|z|} \end{cases}}$

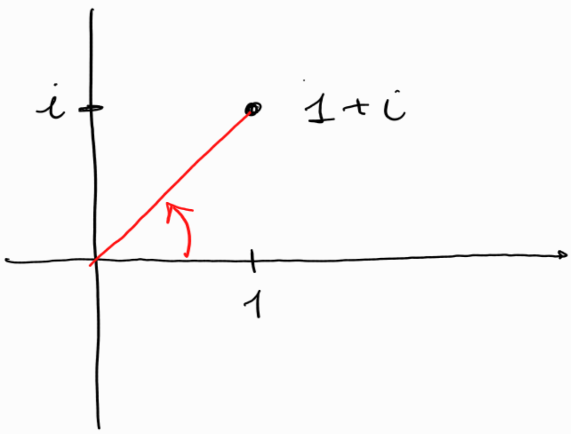
Questo determina φ a meno di multipli di 2π.

<u>OSS</u> se $x+iy=0$, cioè $z=0$, cioè $|z|=0$, φ è indeterminato

$z = 1+i$

$|z| = \sqrt{2}$

$\varphi = \frac{\pi}{4} + 2k\pi \qquad (k\in\mathbb{Z})$



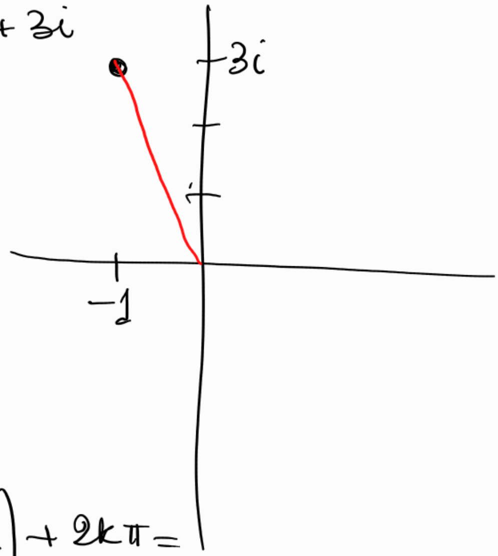
$$\begin{cases} \text{sen}\,\varphi = \frac{1}{\sqrt{2}} \\ \cos\varphi = \frac{1}{\sqrt{2}} \end{cases} \Rightarrow \varphi = \frac{\pi}{4} + 2k\pi$$

$$1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$z = 1+i = \sqrt{2}\cos\frac{\pi}{4} + i\sqrt{2}\sin\frac{\pi}{4} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$z = x + iy = |z|(\cos\varphi + i\sin\varphi)$$

Rappresentare in forma polare $z = -1 + 3i$



$$|z| = \sqrt{1+9} = \sqrt{10}$$

$$\begin{cases} \cos\varphi = -\dfrac{1}{\sqrt{10}} \\ \sin\varphi = \dfrac{3}{\sqrt{10}} \end{cases}$$

$$\varphi = \arccos\left(-\frac{1}{\sqrt{10}}\right) + 2k\pi = \pi - \arcsin\left(\frac{3}{\sqrt{10}}\right) + 2k\pi =$$

$$= \pi - \operatorname{arctg} 3 + 2k\pi.$$

$$-1 + 3i = \sqrt{10}\, e^{i \arccos\left(-\frac{1}{\sqrt{10}}\right)}$$

Moltiplichiamo tra loro due numeri complessi usando coord. polari.

$$z = |z|(\cos\varphi + i\,\text{sen}\,\varphi)$$
$$w = |w|(\cos\theta + i\,\text{sen}\,\theta)$$

$$z \cdot w = |z||w|\Big(\underbrace{\cos\varphi\cos\theta - \text{sen}\,\varphi\,\text{sen}\,\theta}_{\cos(\varphi+\theta)} + i\underbrace{(\cos\varphi\,\text{sen}\,\theta + \text{sen}\,\varphi\cos\theta)}_{\text{sen}(\varphi+\theta)}\Big)$$

$$= \underline{|z||w|}\big(\cos(\varphi+\theta) + i\,\text{sen}(\varphi+\theta)\big)$$

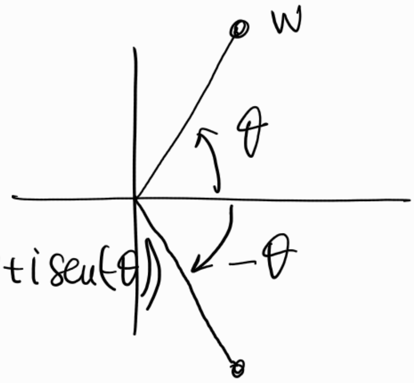
Il prodotto di $z \cdot w$ ha come modulo $|z||w|$, e come argomento (angolo) la somma degli argomenti di z e w.

TEOREMA

1) Facendo il prodotto di due numeri complessi, i moduli si moltiplicano, gli argomenti si sommano.
2) Facendo il rapporto di due numeri complessi, i moduli si dividono tra loro, gli argomenti si sottraggono

$$\frac{z}{w} = \frac{|z|}{|w|}\left(\cos(\varphi-\theta) + i\ \sin(\varphi-\theta)\right)$$

$$z = |z|\,(\cos\varphi + i\,\mathrm{sen}\,\varphi)$$
$$w = |w|\,(\cos\theta + i\,\mathrm{sen}\,\theta) \implies$$
$$\implies \overline{w} = |w|\,(\cos(-\theta) + i\,\mathrm{sen}(-\theta))$$



$$\frac{z}{w} = z \cdot \frac{1}{w} =$$

$$\frac{1}{w} = \frac{\overline{w}}{w\cdot\overline{w}} = \frac{\overline{w}}{|w|^2} = \frac{\cancel{|w|}}{|w|^{\cancel{2}}} \cdot (\cos(-\theta) + i\,\mathrm{sen}(-\theta)) \qquad w\cdot\overline{w} = |w|^2$$

$$\frac{1}{w} = \frac{1}{|w|}\,(\cos(-\theta) + i\ \mathrm{sen}(-\theta))$$

$$z \cdot \frac{1}{w} = |z| \cdot \frac{1}{|w|}\left(\cos(\varphi-\theta) + i\ \mathrm{sen}(\varphi-\theta)\right)$$

applico la parte 1) del Teorema

Notazione esponenziale:

poniamo $\boxed{e^{i\varphi} := \cos\varphi + i\,\mathrm{sen}\,\varphi}$ $\underline{\forall\varphi\in\mathbb{R}}$.

$$z = |z|\,(\cos\varphi + i\ \mathrm{sen}\,\varphi) = |z|\,e^{i\varphi}$$

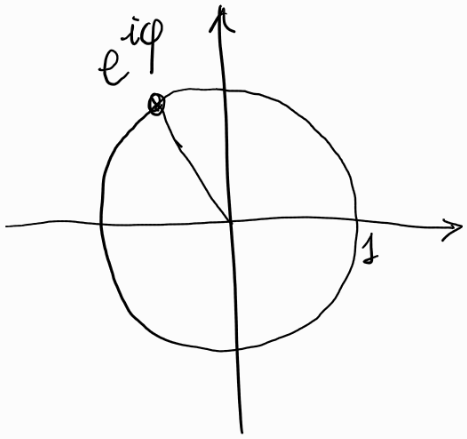
$$z\cdot w = \underbrace{|z|e^{i\varphi}}_{z}\ \underbrace{|w|e^{i\theta}}_{w} = |z|\,|w|\,e^{i(\varphi+\theta)}$$

$$\frac{z}{w} = \frac{|z|\,e^{i\varphi}}{|w|\,e^{i\theta}} = \frac{|z|}{|w|}\,e^{i(\varphi-\theta)}.$$

$$e^{i0} = \cos 0 + i\,\mathrm{sen}\,0 = 1$$

$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\,\mathrm{sen}\,\frac{\pi}{2} = i$$

$$e^{i\pi} = -1 \implies \boxed{e^{i\pi} + 1 = 0}$$ Identità di Eulero.

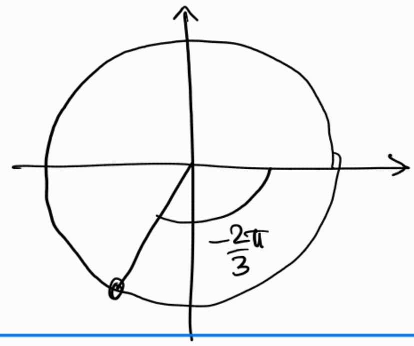


$e^{i\varphi}$ rappresenta i punti della circonferenza unitaria

$$x^2 + y^2 = 1$$

$$e^{-i\frac{\pi}{2}} = -i = e^{i\frac{3}{2}\pi}$$

$$e^{\frac{4}{3}\pi i} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = -\frac{1+\sqrt{3}i}{2}$$



Voglio calcolare $z \cdot w$ e $\frac{z}{w}$, dove $z = 1-i$
$w = 2+2i$

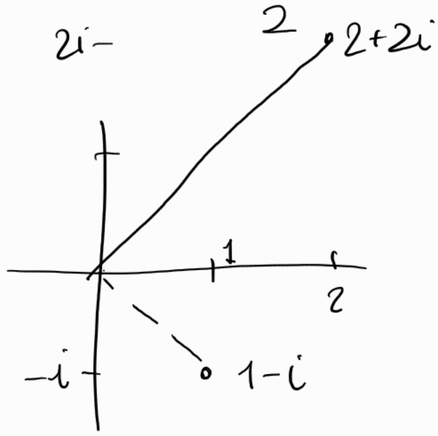
1) in coordinate cartesiane

$$z \cdot w = (1-i)(2+2i) = 2(1-i)(1+i) = 2 \cdot (1+1) = 4$$

$$\frac{z}{w} = \frac{1-i}{2+2i} = \frac{1-i}{2(1+i)} \cdot \frac{(1-i)}{1-i} = \frac{\cancel{1} - \cancel{1} - 2i}{2 \cdot 2} = -\frac{i}{2}.$$

$2i-$

2) in coordinate polari

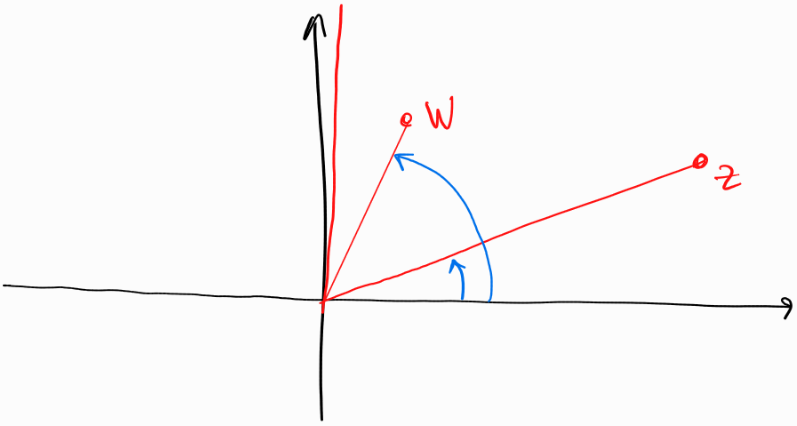


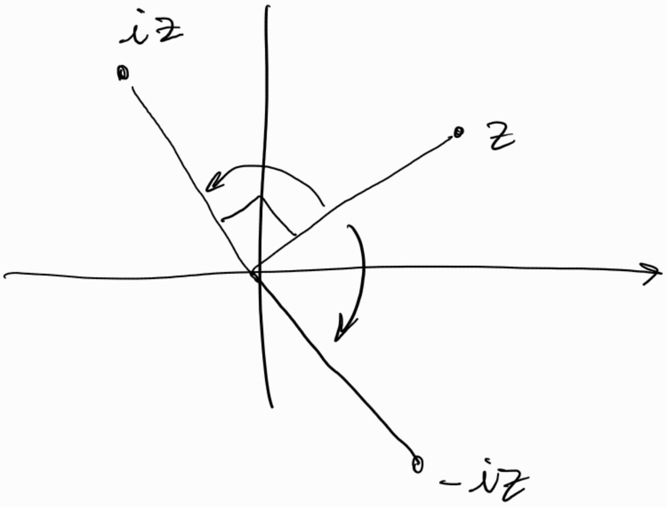
$$z = 1-i = \sqrt{2}\, e^{-i\frac{\pi}{4}}$$

$$w = 2+2i = 2\sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$z \cdot w = 4\, e^{i\left(-\frac{\pi}{4} + \frac{\pi}{4}\right)} = 4$$

$$\frac{z}{w} = \frac{\cancel{\sqrt{2}}}{2\cancel{\sqrt{2}}}\, e^{i\left(-\frac{\pi}{4} - \frac{\pi}{4}\right)} = \frac{1}{2}\, e^{-i\frac{\pi}{2}} = -\frac{i}{2}$$





Calcolare $(1+i)^4$.

a) in coordinate cartesiane

$$(1+i)^4 = 1^4 + 4\cdot 1^3 i + 6\cdot 1^2 \cdot i^2 + 4\cdot 1\, i^3 + i^4 =$$
$$= 1 + \cancel{4i} - 6 - \cancel{4i} + 1 = -4$$

b) in coordinate polari

$$1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$(1+i)^4 = (\sqrt{2})^4 e^{i\pi} = -4$$

$$(1+i)^{20} = (\sqrt{2})^{20} e^{i\frac{20\pi}{4}} = -2^{10} = -1024$$

Potenze di numeri complessi

se $z = |z| e^{i\varphi} \Rightarrow z^n = |z|^n e^{i\varphi n} =$
$(n \in \mathbb{N})$

$= |z|^n \left(\cos(n\varphi) + i \operatorname{sen}(n\varphi) \right)$

$|z| = 1 \Rightarrow (\cos\varphi + i \operatorname{sen}\varphi)^n = \cos(n\varphi) + i \operatorname{sen}(n\varphi)$
formule di De Moivre.

Radici n-esime di un numero complesso

Sia $z \in \mathbb{C}$. Voglio risolvere l'eq^{ne}

$$w^n = z$$

rispetto a w, dove $n \in \mathbb{N}_+$ è fissato.

In altre parole, voglio trovare le radici n-esime di z.

$\underbrace{z = |z| e^{i\varphi}}_{\text{note}}$ $\qquad w = |w| e^{i\theta}$

$\qquad\qquad w^n = |w|^n e^{in\theta}$

Quindi devo risolvere

$$|w|^n e^{in\theta} = |z| e^{i\varphi}$$

$$\Leftrightarrow \begin{cases} |w|^n = |z| \\ n\theta = \varphi + 2k\pi \end{cases} \quad \Leftrightarrow \quad |w| = \sqrt[n]{|z|} \text{ in senso reale} \quad (k \in \mathbb{Z}).$$