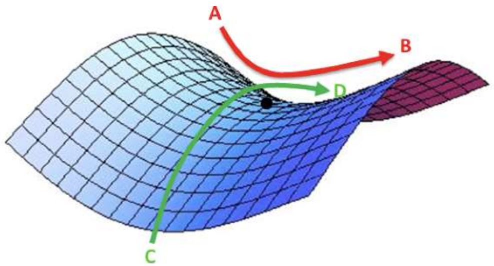
Ph.D. in Information and Communication Engineering

Ph.D. Course on

Analytical Techniques for Wave Phenomena



Lesson 9

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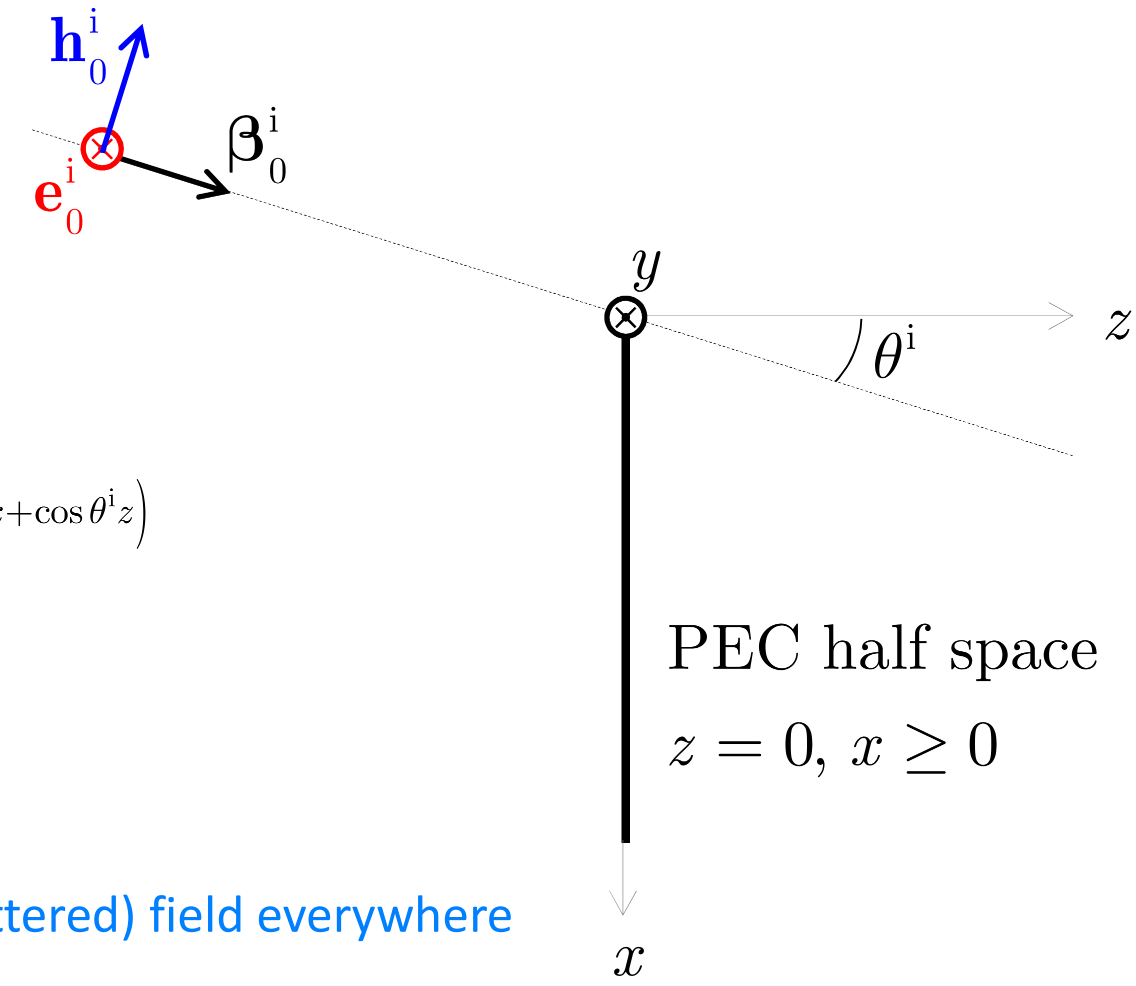
A.y. 2023-2024

The Sommerfeld Half-Plane Problem: Generalities

The Sommerfeld Half-Plane Problem

A uniform plane wave impinges normally on a PEC half plane with the electric field parallel to its rim:



$$
\begin{aligned}
\mathbf{E}^{\mathrm{i}} &= E_0^{\mathrm{i}} \mathbf{e}_0^{\mathrm{i}} e^{-jk_0 \boldsymbol{\beta}_0^{\mathrm{i}} \cdot \mathbf{r}} \\
&= E_0^{\mathrm{i}} \mathbf{y}_0 e^{-jk_0 \left(\sin\theta^{\mathrm{i}} x + \cos\theta^{\mathrm{i}} z \right)}
\end{aligned}
$$

Problem:
find the total (incident + scattered) field everywhere

Some History

- **A. Sommerfeld** provided the exact solution to the problem of plane-wave diffraction by a PEC half plane in 1896.

Mathematische Theorie der Diffraction.

(Mit einer Tafel.)

Von

A. Sommerfeld in Göttingen.

Die Theorie der Diffraction, wie sie von Fresnel begründet und von Kirchhoff analytisch präcisirt ist, genügt aus verschiedenen Gründen nicht den Anforderungen der mathematischen Strenge. Einige

[A. Sommerfeld, "Mathematische theorie der diffraction," *Math. Ann.*, vol. 47, no. 2, pp. 317–374, 1896.]

Such a solution, although obtained with an elaborate approach not amenable to be used in different problems, was historically important as it was *the first exact solution* to a diffraction problem.

Some History

- It was realized in the 1940s that the powerful **Wiener-Hopf approach** could be used to solve the problem.

  [E. T. Copson, "On an integral equation arising in the theory of diffraction," *Quart. J. Math*., vol. 17, pp. 19-34, 1946.]

  [J. F. Carlson and A. E. Heins, "The reflection of an electromagnetic plane wave by an infinite set of plates," *Quart. Appl. Math*., vol. 4, pp. 313–329, 1947.]

- In 1951 P. C. Clemmow proposed an alternative approach (based on **dual integral equations**) that works directly with the spectra of the field quantities.

  [P. C. Clemmow, "A method for the exact solution of a class of two-dimensional diffraction problems," *Proc. Roy. Soc. London*, Ser. A, vol. 205, pp. 286–308, 1951.]

The Wiener-Hopf/dual-integral-equation method employ crucially *complex analysis* and can be used to solve many different diffraction problems (e.g., half planes and junctions with impedance boundary or transition conditions).

Some History

- Much later, in 1983, F. Gori found that an **elementary solution** to the Sommerfeld half-plane problem could be obtained by employing *cylindrical functions of half-integer order*.

Volume 48, number 2 OPTICS COMMUNICATIONS 15 November 1983

DIFFRACTION FROM A HALF-PLANE. A NEW DERIVATION OF THE SOMMERFELD SOLUTION ☆

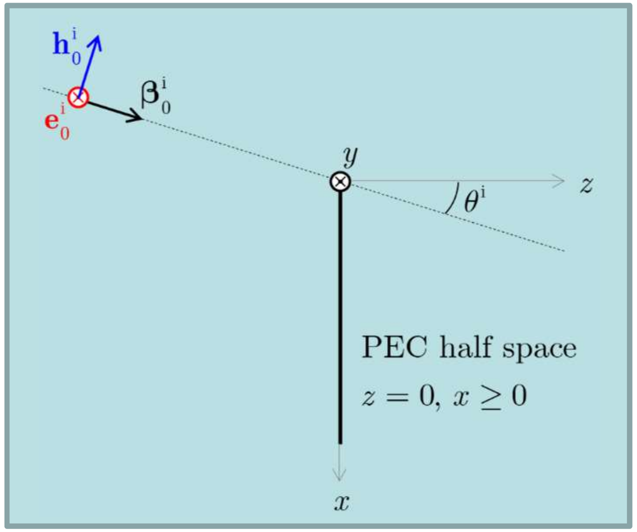
F. GORI
Dipartimento di Fisica, Università di Roma, Roma 00185, Italy

[F. Gori, "Diffraction from a half plane. A new derivation of the Sommerfeld solution," *Optics Commun.*, vol. 48, no. 2, pp. 67–70, Nov. 1983.]

This elementary solution will be illustrated first, after discussing some generalities in the next slides.

Translational Invariance and TM Polarization

- The PEC half plane is **translationally invariant** w.r.t. the y direction
- The incident field is **independent** of the y coordinate and TM^y



- **The scattered (and hence the total) field is also independent of the y coordinate and TM^y**
- **The only nonzero components of the fields are**

$$E_y, H_x, H_z$$

- **From the Maxwell equations we may express the magnetic field in terms of the electric field as**

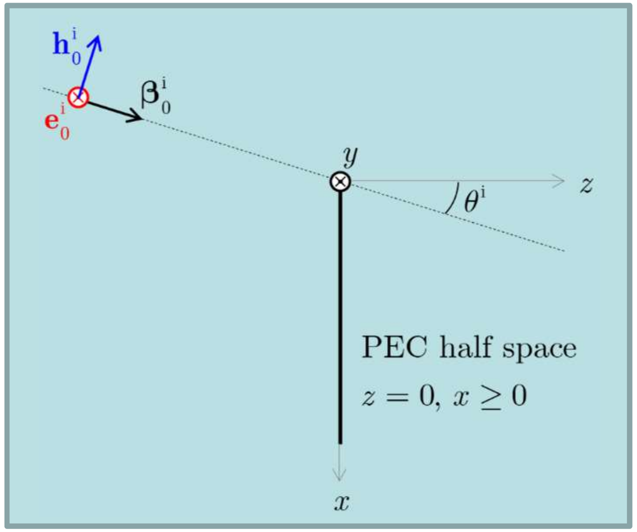
$$H_x = \frac{1}{j\omega\mu_0}\frac{\partial E_y}{\partial z}, \qquad H_z = -\frac{1}{j\omega\mu_0}\frac{\partial E_y}{\partial x}$$

Electric Current Density on the PEC Half Plane

$$\mathbf{J}_s = \mathbf{z}_0 \times \left[\mathbf{H}\right]_-^+ = \mathbf{z}_0 \times \left[\mathbf{x}_0 H_x + \mathbf{z}_0 H_z\right]_-^+ = \mathbf{y}_0 \left[H_x\right]_-^+$$

$$= \mathbf{y}_0 \frac{1}{j\omega\mu_0} \left[\frac{\partial E_y}{\partial z}\right]_-^+ = \mathbf{y}_0 \frac{1}{j\omega\mu_0} \left[\frac{\partial E_y^s}{\partial z}\right]_-^+$$

the incident field and its derivatives are continuous in the entire space



The normal derivative of the scattered electric field is **discontinuous** across the PEC half plane.

Such a discontinuity is related to the existence of an electric current density, which flows **parallel to the rim** of the PEC half plane

Scalar Nature of the Problem

The problem can thus be reduced to a **scalar one** in terms of the scattered electric field.

By letting $\mathcal{D} = \left\{ z = 0, x \geq 0 \right\}$ be the PEC half plane, we have:

$$\nabla^2_{xz} E^{\mathrm{s}}_y + k^2_0 E^{\mathrm{s}}_y = 0, \ \text{ in } \mathbb{R}^2_{xz} \setminus \mathcal{D}$$

2D Helmholtz equation

$$E^{\mathrm{s}}_y = -E^{\mathrm{i}}_y = -E^{\mathrm{i}}_0 e^{-jk_0 \sin\theta^{\mathrm{i}} x}, \text{ on } \mathcal{D}$$

boundary condition
on the PEC half plane

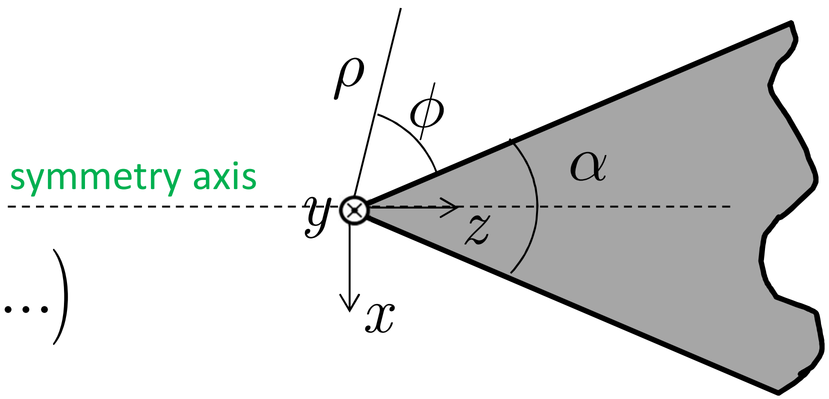
Edge Condition

To ensure the uniqueness of the solution, the correct behavior of the field quantities in the vicinity of the **geometrical singularities** (edges, vertices) of the boundary must be specified.

Consider for instance a PEC wedge with internal angle α placed inside a uniform dielectric medium:

$$E_y \sim \rho^{\nu}\left(a_0 + a_1\rho + \ldots\right)$$

$$\Rightarrow H_x, H_z \sim \rho^{\nu-1}\left(b_0 + b_1\rho + \ldots\right)$$



Generally, the behavior of an electromagnetic field in the neighborhood of the common edge of angular dielectric or conducting regions is determined from the condition that **the energy density must be integrable over any finite domain** (the so-called **edge condition**).

[J. Meixner, “Die Kantenbedingung in der Theorie der Beugung electromagnetischer Wellen an vollkommen leitenden ebenen Schirmen," *Ann. Phys.*, vol. 6, pp. 1-9, 1949.]

Edge Condition

By further enforcing the boundary conditions on the faces of the PEC wedge, the exponent ν can be determined as

$$\nu = \frac{\pi}{2\pi - \alpha}$$

Therefore, in the case of a PEC half plane, equivalent to a PEC wedge with zero internal angle, it results $\nu = 1/2$:

$$E_y \sim \rho^{1/2}$$

$$H_x, H_z, J_z \sim \rho^{-1/2}$$

$$\rho = \sqrt{x^2 + z^2} \rightarrow 0$$

Symmetry of the Scattered Field

- The scattered field is produced by the current density on the PEC half plane, which is purely tangential.
- The plane $z = 0$ is a symmetry plane for the scattered field.

$$\Downarrow$$

The scattered field is **even-symmetric** w.r.t. the $z{=}0$ plane:

$$E_y^{\mathrm{s}}\left(x,-z\right) = E_y^{\mathrm{s}}\left(x,z\right)$$

hence it is sufficient to find the scattered field in the half space $z \geq 0$

Diffraction = Interaction + Propagation

As any problem of diffraction by a screen, also the present one can be decomposed into two steps:

1) ***<u>Interaction problem</u>***

Find the scattered field on the plane $z = 0$. In the present case, since the scattered field is known on the PEC half plane, this reduces to determining the scattered field on the half plane $z = 0$, $x < 0$.

2) ***<u>Propagation Problem</u>***

Determine the field in the half space $z > 0$ from knowledge of the field on the plane $z = 0$ (the so-called 'aperture field')

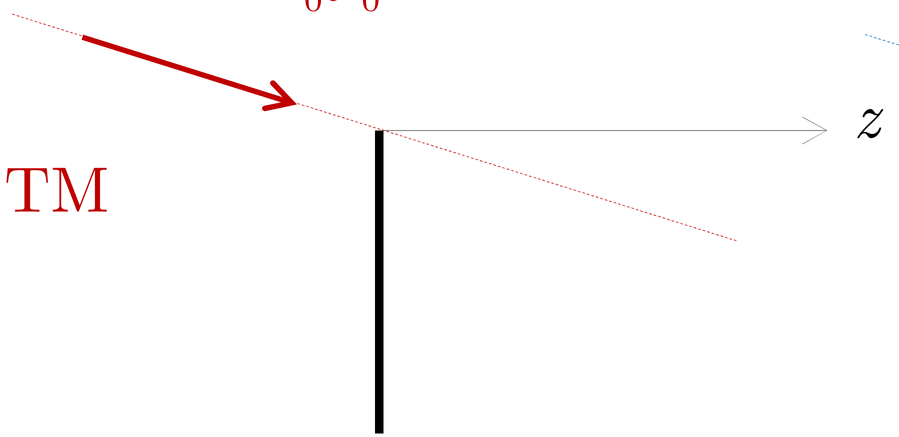
<u>Note:</u> the difficult part is 1); in fact, 2) can be solved by expressing the field in the half space $z > 0$ through standard radiation integrals in terms of the aperture field.

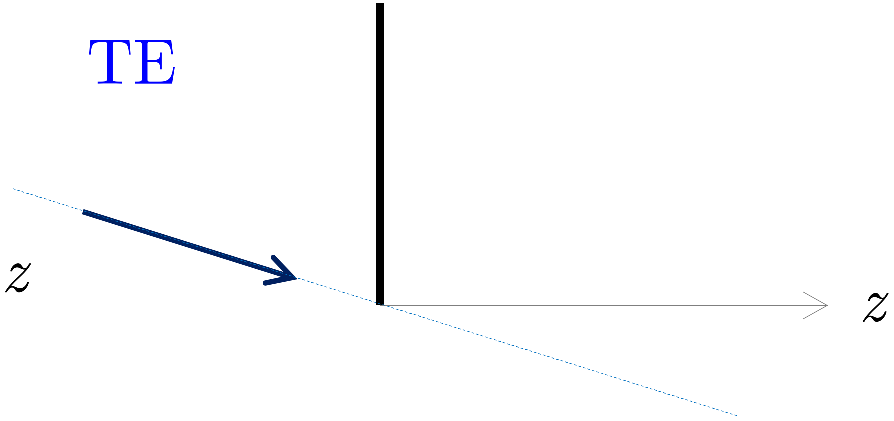
Generalizations: TE Incidence

It is important to observe that, once a solution has been found, it can be immediately generalized in different directions.

First, the case of TE incidence can be obtained by using the **Babinet principle** in its full electromagnetic form:

$$\mathbf{E}^{\mathrm{i,TM}} = E_0^{\mathrm{i}} \mathbf{y}_0 e^{-jk_0\left(\sin\theta^{\mathrm{i}} x + \cos\theta^{\mathrm{i}} z\right)}$$





$$\mathbf{H}^{\mathrm{i,TE}} = -\frac{E_0^{\mathrm{i}}}{\zeta_0} \mathbf{y}_0 e^{-jk_0\left(\sin\theta^{\mathrm{i}} x + \cos\theta^{\mathrm{i}} z\right)}$$

$$\boxed{\mathbf{E}^{\mathrm{TE}} = \zeta_0 \left(\mathbf{H}^{\mathrm{i,TM}} - \mathbf{H}^{\mathrm{TM}} \right)} \quad z \geq 0$$

Generalizations: Oblique Incidence

Let us now consider an incident plane wave propagating at an angle ψ^{i} w.r.t. the y axis:

$$\mathbf{E}^{\mathrm{i,TM}} = \mathbf{y}_0 e^{-jk_0 \cos\psi^{\mathrm{i}} y} \underbrace{e^{-jk_0 \sin\psi^{\mathrm{i}}\left(\sin\theta^{\mathrm{i}} x + \cos\theta^{\mathrm{i}} z\right)}}_{\doteq e_y^{\mathrm{i}}\left(x,z;k_0,\psi^{\mathrm{i}},\theta^{\mathrm{i}}\right)}$$

(the case of normal incidence being recovered by letting $\psi^{\mathrm{i}} = \pi/2$).

Since this is an **eigenfunction** of any **translation operator** along the y axis and since the structure is **translationally invariant** along the same axis, we conclude that the scattered field must be of the form:

$$\mathbf{E}^{\mathrm{s,TM}} = \mathbf{y}_0 e^{-jk_0 \cos\psi^{\mathrm{i}} y} e_y^{\mathrm{s}}\left(x, z; k_0, \psi^{\mathrm{i}}, \theta^{\mathrm{i}}\right)$$

Generalizations: Oblique Incidence

normal incidence

$$\nabla^2_{xz} e^{\mathrm{s}}_y + k_0^2 e^{\mathrm{s}}_y = 0, \ \text{ in } \mathbb{R}^2_{xz} \setminus \mathcal{D}$$

$$e^{\mathrm{s}}_y = -E^{\mathrm{i}}_0 e^{-jk_0 \sin\theta^{\mathrm{i}} x}, \ \text{ on } \mathcal{D}$$

oblique incidence

$$\nabla^2_{xz} e^{\mathrm{s}}_y + k_{\mathrm{t}}^2 e^{\mathrm{s}}_y = 0, \ \text{ in } \mathbb{R}^2_{xz} \setminus \mathcal{D}$$

$$e^{\mathrm{s}}_y = -E^{\mathrm{i}}_0 e^{-jk_{\mathrm{t}} \sin\theta^{\mathrm{i}} x}, \ \text{ on } \mathcal{D}$$

$$k_{\mathrm{t}} = k_0 \sin\psi^{\mathrm{i}}$$

$$e^{\mathrm{s}}_y\left(x, z; k_0, \psi^{\mathrm{i}}, \theta^{\mathrm{i}}\right) = e^{\mathrm{s}}_y\left(x, z; k_0' = k_0 \sin\psi^{\mathrm{i}}, \frac{\pi}{2}, \theta^{\mathrm{i}}\right)$$

The scattered field for oblique incidence can be obtained from the scattered field for normal incidence calculated at the **scaled frequency** $\omega' = \omega \sin\psi^{\mathrm{i}}$

The Sommerfeld Half-Plane Problem: Elementary Solution

Cylindrical Waves of Half-Integer Order

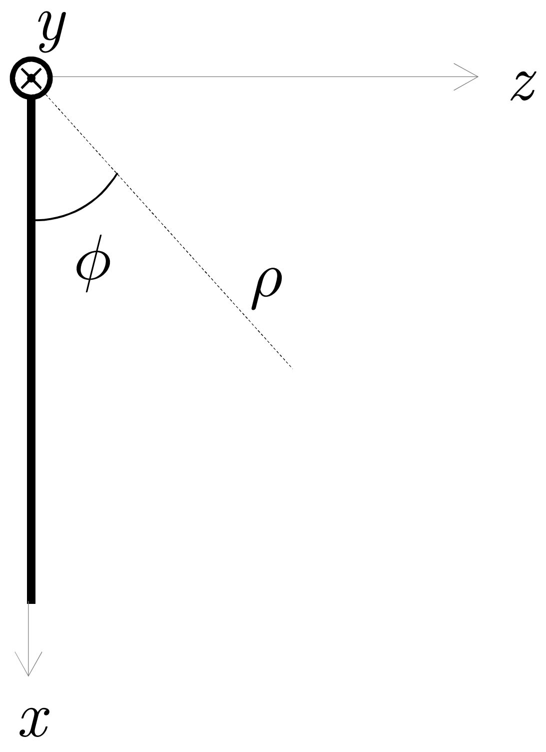
The interaction problem can be solved using a superposition of **elementary solutions of the Helmholtz equation** which correspond to fields that could be **generated by currents flowing only in the half plane** $x \geq 0$, $z = 0$.

[F. Gori, "Diffraction from a half plane. A new derivation of the Sommerfeld solution," *Optics Commun.*, vol. 48, no. 2, pp. 67–70, Nov. 1983.]

Consider the electric field

$$E_y = E_0 \frac{e^{-jk_0\rho}}{\underbrace{\sqrt{k_0\rho}}_{=\sqrt{\frac{\pi}{2}}H_{1/2}(k_0\rho)}} \sin\left(\frac{\phi}{2}\right)$$

with $0 \leq \phi \leq 2\pi$



This is the electric field of an **anisotropic cylindrical wave** of order $n = 1/2$.

Cylindrical Waves of Half-Integer Order

The relevant magnetic field is:

$$\boxed{\mathbf{H} = -\frac{1}{j\omega\mu_0}\nabla\times\mathbf{E} = -\frac{1}{jk_0\zeta_0}\left[\boldsymbol{\rho}_0\frac{1}{\rho}\frac{\partial E_y}{\partial\phi} - \boldsymbol{\phi}_0\frac{\partial E_y}{\partial\rho}\right]}$$

$$H_\rho = j\frac{E_0}{\zeta_0}\frac{e^{-jk_0\rho}}{\left(k_0\rho\right)^{3/2}}\frac{1}{2}\cos\left(\frac{\phi}{2}\right)$$

discontinuous on $\phi = 0,\ 2\pi$

$$H_\phi = -j\frac{E_0}{\zeta_0}\frac{1+2jk_0\rho}{\left(k_0\rho\right)^{3/2}}e^{-jk\rho}\frac{1}{2}\sin\left(\frac{\phi}{2}\right)$$

continuous everywhere

Cylindrical Waves of Half-Integer Order

We deduce that the field of this cylindrical wave is produced by a y-directed surface current which is **nonzero only on the half plane** $\phi = 0$:

$$\boxed{J_y = \left[H_\rho \right]_{\phi=2\pi}^{\phi=0} = j \frac{E_0}{\zeta_0} \frac{e^{-jk_0\rho}}{\left(k_0\rho\right)^{3/2}}, \quad \text{on} \quad \phi = 0}$$

In order to solve the interaction problem, we will look for a **superposition** of such cylindrical waves, with their axes (parallel to one another and to the y-plane) lying on the half-plane $x \geq 0$, $z = 0$.

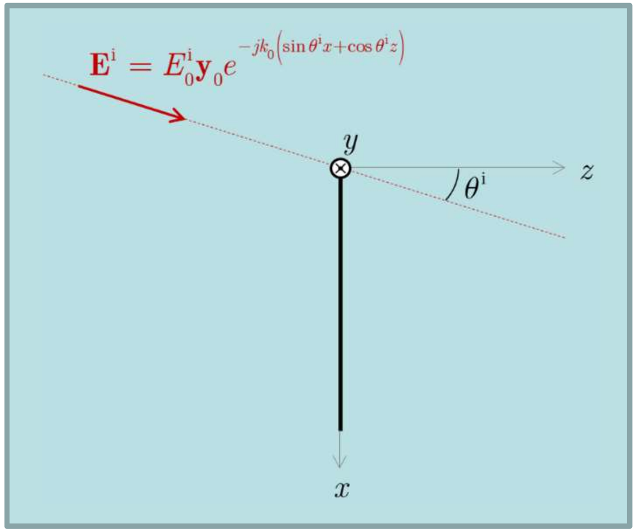
Enforcing the Boundary Condition

Since E_y vanishes for $\phi = 0$, the electric field at any point on the x-axis depends only on the waves originating on the right of that point.

Therefore, we may write:

$$E_y^{\mathrm{s}}\left(x, z = 0\right) = \int_x^{+\infty} p\left(\xi\right) \frac{e^{-jk_0\left(\xi - x\right)}}{\sqrt{k_0\left(\xi - x\right)}} \mathrm{d}\xi,$$

$$x \geq 0$$



The function $p(\xi)$ must be such that the scattered field be equal and opposite to the incident field for any $x \geq 0$. This allows us for obtaining an integral equation with the function $p(\xi)$ as an unknown…

Enforcing the Boundary Condition

$$\boxed{\int_x^{+\infty} p\left(\xi\right)\frac{e^{-jk_0\left(\xi-x\right)}}{\sqrt{k_0\left(\xi-x\right)}}\mathrm{d}\xi = -E_0^{\mathrm{i}}e^{-jk_0\sin\theta^{\mathrm{i}}x}, \qquad x\geq 0}$$

By a simple change of variable we have:

$$\int_0^{+\infty} p\left(\xi'+x\right)\frac{e^{-jk_0\xi'}}{\sqrt{k_0\xi'}}\mathrm{d}\xi' = -E_0^{\mathrm{i}}e^{-jk_0\sin\theta^{\mathrm{i}}x}, \qquad x\geq 0$$

whose solution is found by inspection to simply be an **exponential function**:

$$p\left(\xi\right) = -CE_0^{\mathrm{i}}e^{-jk_0\sin\theta^{\mathrm{i}}\xi} \;\Rightarrow\; p\left(\xi'+x\right) = -CE_0^{\mathrm{i}}e^{-jk_0\sin\theta^{\mathrm{i}}x}e^{-jk_0\sin\theta^{\mathrm{i}}\xi'}$$

Enforcing the Boundary Condition

By inserting this into the equation, we may find the value of the constant C :

$$C\int_0^{+\infty} e^{-jk_0 \sin\theta^{\mathrm{i}}\xi'} \frac{e^{-jk_0\xi'}}{\sqrt{k_0\xi'}}\,\mathrm{d}\xi' = 1$$

whence, by letting $\mu^2 = k_0\xi'(1+\sin\theta^{\mathrm{i}})$

$$\frac{2C}{k_0\sqrt{1+\sin\theta^{\mathrm{i}}}} \underbrace{\int_0^{+\infty} e^{-j\mu^2}\,\mathrm{d}\mu}_{=e^{-j\pi/4}\sqrt{\pi}/2} = 1$$

(complete Fresnel integral)

$$C = \frac{k_0}{\sqrt{\pi}} e^{+j\pi/4}\sqrt{1+\sin\theta^{\mathrm{i}}}$$

Solution of the Interaction Problem

The scattered field on the entire aperture plane $z = 0$ is finally

$$\boxed{E_y^{\mathrm{s}}\left(x, z=0\right) = \begin{cases} -E_0^{\mathrm{i}} e^{-jk_0 \sin\theta^{\mathrm{i}} x}, & x \geq 0 \\ -\dfrac{2E_0^{\mathrm{i}}}{\sqrt{\pi}} e^{+j\pi/4} F\left(\sqrt{k_0 \left|x\right| \left(1 + \sin\theta^{\mathrm{i}}\right)}\right), & x < 0 \end{cases}}$$

where

$$F\left(t\right) = \int_t^{+\infty} e^{-j\mu^2} \mathrm{d}\mu$$

is a complex Fresnel integral.

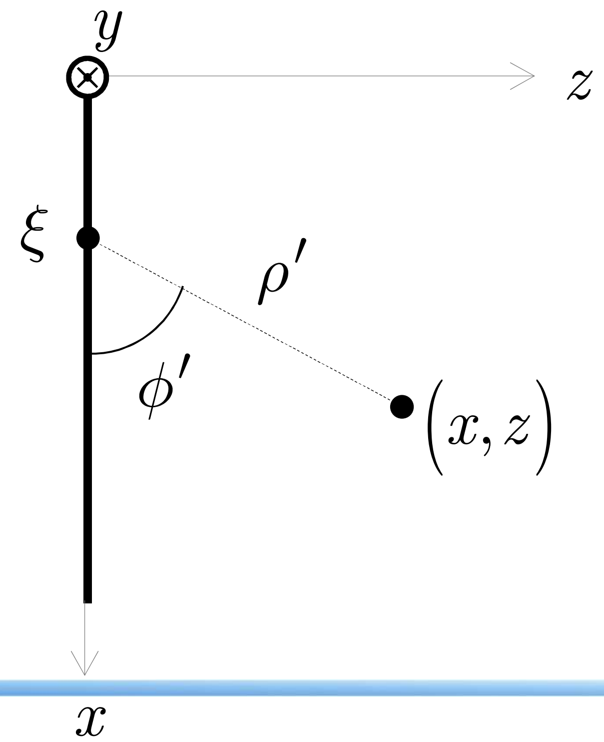
Solution of the Propagation Problem

The scattered field in the half space $z > 0$ can be calculated with any propagation formula like the Rayleigh-Sommerfeld or the plane-wave expansion formulas.

Alternatively, the scattered field at any point can be evaluated directly as a superposition of the half-integer cylindrical waves employed to solve the interaction problem:

$$\boxed{E_y^{\mathrm{s}}\left(x, z \geq 0\right) = \int_0^{+\infty} p\left(\xi\right) \frac{e^{-jk_0\rho'}}{\sqrt{k_0\rho'}} \sin\frac{\phi'}{2}\, \mathrm{d}\xi}$$

$$\rho' = \sqrt{\left(x-\xi\right)^2 + z^2} \qquad \tan\phi' = \frac{z}{x-\xi}$$

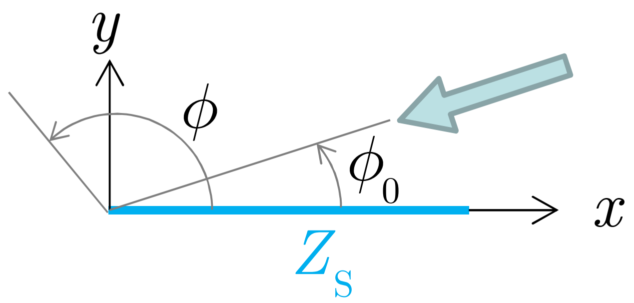


The Sommerfeld Half-Plane Problem:
Dual Integral Equations/Wiener-Hopf Method

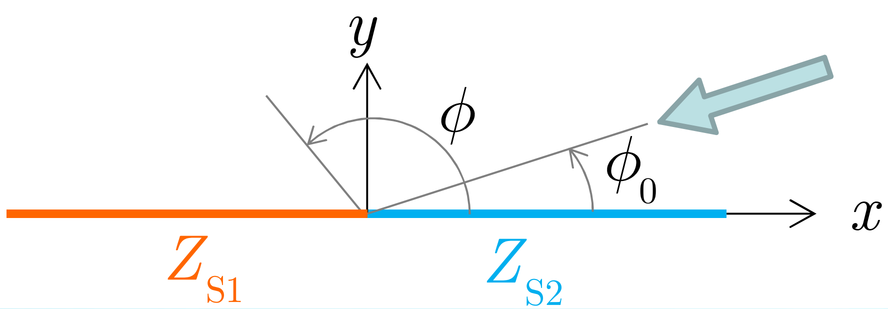
The Wiener-Hopf Method

The analysis of the half-plane diffraction problem is generally carried out with the **Wiener-Hopf method**, which allows for generalizing the solution to

- **half planes with impedance boundary/transition conditions:**



- **two-part impedance planes:**



The Wiener-Hopf Method

The analytical solution of the integral equation that results from the application of the boundary conditions is obtained in the Wiener-Hopf method by:

1) **taking the Fourier transform** of the integral equation, thereby

2) **reducing it to a functional equation**, which is in turn solved by invoking properties of analytic functions (crucially, the generalized *Liouville theorem* for entire functions).

[https://en.wikipedia.org/wiki/Liouville%27s_theorem_(complex_analysis)]

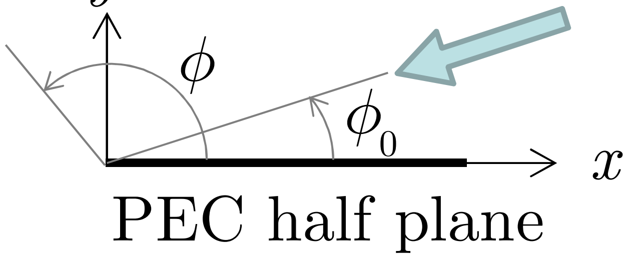
Since the crucial (and most difficult) step of the method consists in **factorizing** a given function into the product of **functions analytic (and without zeroes) in the upper and lower halves of the complex plane**, the method is also known (especially in the Soviet literature) as the **Factorization Method**.

Wiener-Hopf Method vs. Dual Integral Equations

A mathematically equivalent but more direct approach, that works directly with the spectra of the field quantities, was proposed in 1951 by Clemmow and is based on the so-called **dual integral equations**.

Since this requires fewer steps to arrive at the solution, it will be adopted here to illustrate the solution for the case of a PEC half plane, TM polarization:

$$\mathbf{E}^{\mathrm{i,TM}} = E_0^{\mathrm{i}}\mathbf{z}_0 e^{jk_0\rho\cos\left(\phi-\phi_0\right)}$$



[T. B. A. Senior and J. L. Volakis, *Approximate boundary conditions in electromagnetics*. London, UK: The IEE, 1995, ch. 3.]

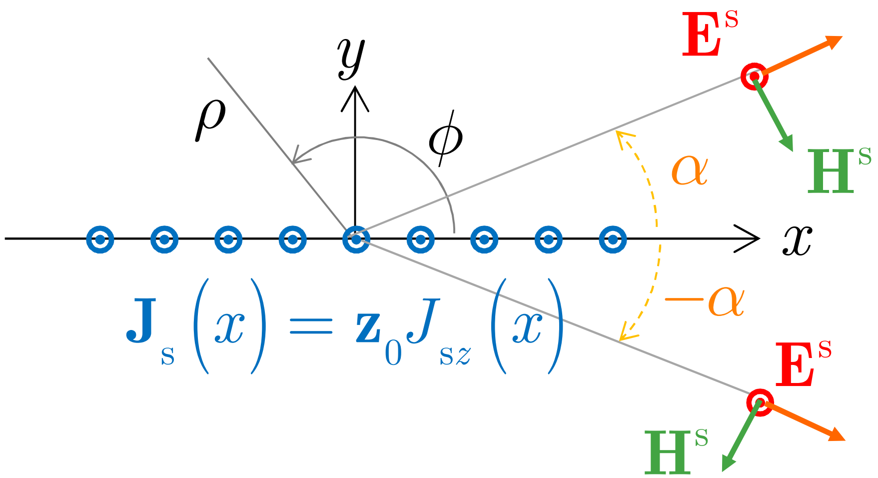
Angular Spectrum Representation of the Fields

A key feature of the dual-integral-equation method is the *a-priori* introduction of the **angular spectrum** $P_\mathrm{e}(\cos\alpha)$ of the scattered field:

$$E_z^\mathrm{s}\left(\rho,\phi\right) = E_0^\mathrm{i} \int\limits_C P_\mathrm{e}\left(\cos\alpha\right) e^{-jk_0\rho\cos\left(\phi\pm\alpha\right)} \mathrm{d}\alpha, \quad y \gtrless 0$$

$$H_x^\mathrm{s}\left(\rho,\phi\right) = \pm\frac{E_0^\mathrm{i}}{\zeta_0} \int\limits_C \sin\alpha P_\mathrm{e}\left(\cos\alpha\right) e^{-jk_0\rho\cos\left(\phi\pm\alpha\right)} \mathrm{d}\alpha, \quad y \gtrless 0$$

valid for fields due to a z-directed electric current distribution in the $y = 0$ plane:



Angular Spectrum Representation of the Fields

To prove the validity of the angular-spectrum representation let us consider the field radiated by the surface current:

$$E_z^{\mathrm{s}}\left(x,y\right) = -j\omega\mu_0 \int_{-\infty}^{+\infty} J_{\mathrm{s}z}\left(x'\right) \frac{1}{4j} H_0^{(2)}\left(k_0\sqrt{\left(x-x'\right)^2+y^2}\right)\mathrm{d}x'$$

(Green's function for the 2D scalar Helmholtz equation)

By letting $y=0$:

$$\begin{aligned} E_z^{\mathrm{s}}\left(x,0\right) &= -j\omega\mu_0 \int_{-\infty}^{+\infty} J_{\mathrm{s}z}\left(x'\right) \frac{1}{4j} H_0^{(2)}\left(k_0\left|x-x'\right|\right)\mathrm{d}x' \\ &= -j\omega\mu_0 J_{\mathrm{s}z}\left(x\right) \otimes \frac{1}{4j} H_0^{(2)}\left(k_0\left|x\right|\right) \end{aligned}$$

Angular Spectrum Representation of the Fields

By Fourier-transforming w.r.t. x : $\tilde{E}_z^{\mathrm{s}}\left(k_x, 0\right) = -j\omega\mu_0 \tilde{J}_{\mathrm{s}z}\left(k_x\right)\dfrac{1}{2jk_y}$

where $k_y = \begin{cases} \sqrt{k_0^2 - k_x^2}, & k_x \le k_0 \\ -j\sqrt{k_x^2 - k_0^2}, & k_x \ge k_0 \end{cases}$

hence

$$\tilde{E}_z^{\mathrm{s}}\left(k_x, y\right) = -j\omega\mu_0 \tilde{J}_{\mathrm{s}z}\left(k_x\right)\frac{e^{-jk_y\left|y\right|}}{2jk_y}$$

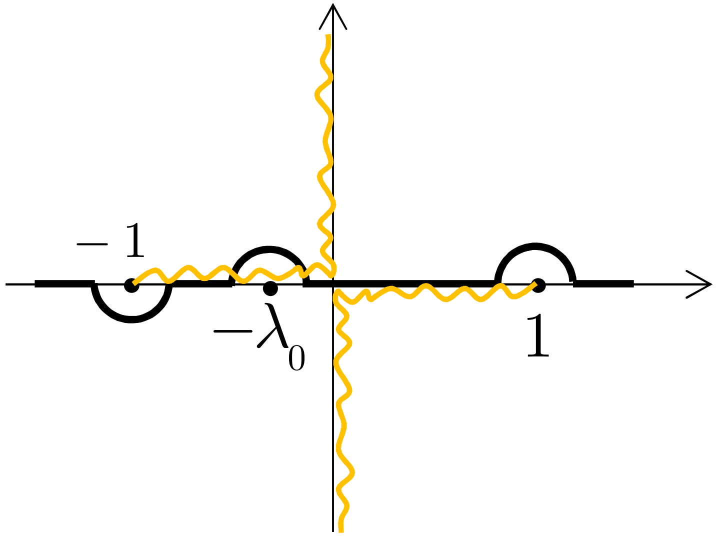
and finally, by inverse Fourier-transforming:

$$E_z^{\mathrm{s}}\left(x, y\right) = -\frac{\omega\mu_0}{4\pi}\int_{-\infty}^{+\infty} \tilde{J}_{\mathrm{s}z}\left(k_x\right)\frac{e^{-jk_y\left|y\right|}}{k_y} e^{-jk_x x}\,\mathrm{d}k_x$$

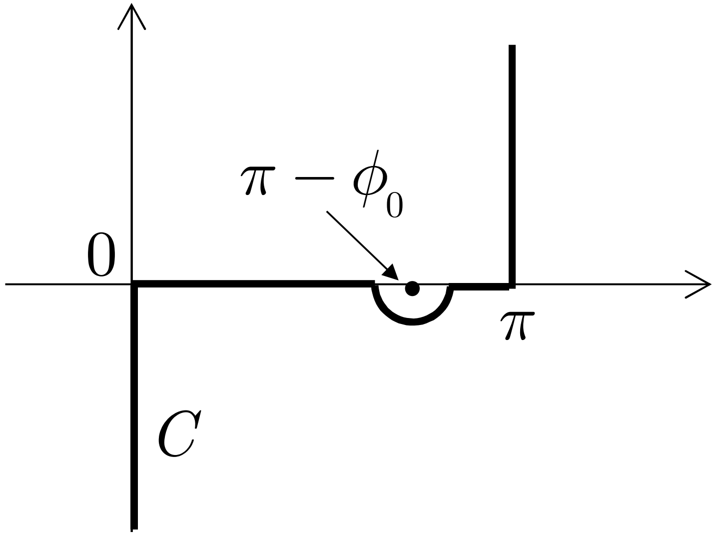
Angular Spectrum Representation of the Fields

We now make the change of variable:

$$k_x = k_0\lambda = k_0 \cos\alpha, \qquad k_y = k_0\sqrt{1-\lambda^2} = k_0 \sin\alpha$$



complex λ-plane



complex α-plane

(the origin and meaning of the indicated pole will be clarified in subsequent slides)

Angular Spectrum Representation of the Fields

This leads to the postulated angular-spectrum representation:

$$E_z^{\mathrm{s}}\left(x,y\right) = -\frac{\omega\mu_0}{4\pi}\int_C \tilde{J}_{\mathrm{s}z}\left(k_0\cos\alpha\right)e^{-jk_0\left(x\cos\alpha+\left|y\right|\sin\alpha\right)}\mathrm{d}\alpha$$

$$E_z^{\mathrm{s}}\left(\rho,\phi\right) = \int_C \underbrace{-\frac{\omega\mu_0}{4\pi}\tilde{J}_{\mathrm{s}z}\left(k_0\cos\alpha\right)}_{=E_0^{\mathrm{i}}P_{\mathrm{e}}\left(\cos\alpha\right)}e^{-jk_0\rho\cos\left(\phi\mp\alpha\right)}\mathrm{d}\alpha,\ \ y \gtrless 0$$

$$H_x^{\mathrm{s}}\left(\rho,\phi\right) = \pm\frac{E_0^{\mathrm{i}}}{\zeta_0}\int_C \sin\alpha\,P_{\mathrm{e}}\left(\cos\alpha\right)e^{-jk_0\rho\cos\left(\phi\mp\alpha\right)}\mathrm{d}\alpha,\ \ y \gtrless 0$$

Spectral Representation of the Fields

Alternatively, using the normalized spectral variable λ, we have the spectral representation of the fields:

$$E_z^{\mathrm{s}}\left(x,y\right)=E_0^{\mathrm{i}}\int_{-\infty}^{+\infty}\frac{P_{\mathrm{e}}\left(\lambda\right)}{\sqrt{1-\lambda^2}}e^{-jk_0x\lambda}e^{-jk_0\left|y\right|\sqrt{1-\lambda^2}}\,\mathrm{d}\lambda$$

$$H_x^{\mathrm{s}}\left(x,y\right)=\pm\frac{E_0^{\mathrm{i}}}{\zeta_0}\int_{-\infty}^{+\infty}P_{\mathrm{e}}\left(\lambda\right)e^{-jk_0x\lambda}e^{-jk_0\left|y\right|\sqrt{1-\lambda^2}}\,\mathrm{d}\lambda$$

Boundary Condition for the Electric Field

Let us write the electric field on the plane $y = 0$ using the λ variable:

$$E_z^{\mathrm{s}}\left(x, y = 0\right) = E_0^{\mathrm{i}} \int_{-\infty}^{+\infty} \frac{P_{\mathrm{e}}\left(\lambda\right)}{\sqrt{1-\lambda^2}} e^{-jk_0 x\lambda} \mathrm{d}\lambda$$

The boundary condition on the PEC plane is: $E_z^{\mathrm{s}}\left(x, y = 0\right) + E_z^{\mathrm{i}}\left(x, y = 0\right) = 0$

$$\Rightarrow \quad \boxed{\int_{-\infty}^{+\infty} \frac{P_{\mathrm{e}}\left(\lambda\right)}{\sqrt{1-\lambda^2}} e^{-jk_0 x\lambda} \mathrm{d}\lambda = -e^{jk_0 x \cos\phi_0}, \qquad x > 0}$$

Boundary Condition for the Magnetic Field

From the spectral representation of the magnetic field we find that it is an **odd function** of y (as expected, by symmetry). In particular, on the plane $y = 0$ we have:

$$H_x^{\mathrm{s}}\left(x, y = 0^{+}\right) = \frac{E_0^{\mathrm{i}}}{\zeta_0} \int_{-\infty}^{+\infty} P_{\mathrm{e}}\left(\lambda\right) e^{-jk_0 x\lambda} \mathrm{d}\lambda = -H_x^{\mathrm{s}}\left(x, y = 0^{-}\right)$$

On the other hand, from the boundary condition for the magnetic field:

$$H_x^{\mathrm{s}}\left(x, y = 0^{+}\right) - H_x^{\mathrm{s}}\left(x, y = 0^{-}\right) = -J_{\mathrm{s}z}\left(x\right)$$

and the fact that the electric current density exists only on $y = 0$, $x > 0$:

$$H_x^{\mathrm{s}}\left(x, y = 0^{+}\right) = H_x^{\mathrm{s}}\left(x, y = 0^{-}\right), \quad x < 0$$

$$\Rightarrow \quad \boxed{\int_{-\infty}^{+\infty} P_{\mathrm{e}}\left(\lambda\right) e^{-jk_0 x\lambda} \mathrm{d}\lambda = 0, \qquad x < 0}$$

Support of the Electric Current Density

We may arrive at the same integral equation by recalling that the spectrum $P_{\mathrm{e}}(\lambda)$ is proportional to the spectral electric surface current density:

$$P_{\mathrm{e}}\left(\lambda\right) = -\frac{\omega\mu_0}{4\pi E_0^{\mathrm{i}}}\tilde{J}_{\mathrm{s}z}\left(k_0\lambda\right)$$

hence by inverse Fourier-transforming:

$$J_{\mathrm{s}z}\left(x\right) = -\frac{2E_0^{\mathrm{i}}}{\omega\mu_0}\int_{-\infty}^{+\infty} P_{\mathrm{e}}\left(\lambda\right) e^{-jk_0x\lambda}\mathrm{d}\lambda$$

and, since the electric current density is nonzero only on the PEC half plane $y = 0$, $x > 0$, we again find:

$$\boxed{\int_{-\infty}^{+\infty} P_{\mathrm{e}}\left(\lambda\right) e^{-jk_0x\lambda}\mathrm{d}\lambda = 0, \qquad x < 0}$$

Dual Integral Equations

Let us collect the two equations thus obtained:

$$\int_{-\infty}^{+\infty} \frac{P_{\mathrm{e}}(\lambda)}{\sqrt{1-\lambda^2}} e^{-jk_0 x\lambda} \mathrm{d}\lambda = -e^{jk_0 x \cos\phi_0}, \qquad x > 0$$

$$\int_{-\infty}^{+\infty} P_{\mathrm{e}}(\lambda) e^{-jk_0 x\lambda} \mathrm{d}\lambda = 0, \qquad x < 0$$

Integral equations of this kind, sharing the **same unknown** but having **different kernels** and **different domains of validity** (in this case, the half lines $x < 0$, $x > 0$) are commonly referred to as **dual integral equations**.

In what follows, they will be solved analytically by means of complex analysis…

Analiticity of the Spectrum in the Upper Plane

Let us recall the connection of the spectrum $P_{\mathrm{e}}(\lambda)$ to the spectral electric current density:

$$P_{\mathrm{e}}\left(\lambda\right) = -\frac{\omega\mu_0}{4\pi E_0^{\mathrm{i}}}\tilde{J}_{\mathrm{s}z}\left(k_0\lambda\right) = -\frac{\omega\mu_0}{4\pi E_0^{\mathrm{i}}}\int_{-\infty}^{+\infty} J_{\mathrm{s}z}\left(x\right)e^{+jk_0x\lambda}\mathrm{d}x$$

$$= -\frac{\omega\mu_0}{4\pi E_0^{\mathrm{i}}}\int_{0}^{+\infty} J_{\mathrm{s}z}\left(x\right)e^{+jk_0x\lambda}\mathrm{d}x$$

and let us integrate this along an **arbitrary closed contour** Γ in the upper λ-plane:

$$\oint_{\Gamma} P_{\mathrm{e}}\left(\lambda\right)\mathrm{d}\lambda = -\frac{\omega\mu_0}{4\pi E_0^{\mathrm{i}}}\oint_{\Gamma}\int_{0}^{+\infty} J_{\mathrm{s}z}\left(x\right)e^{+jk_0x\lambda}\mathrm{d}x\,\mathrm{d}\lambda$$

Thanks to the fact that the inner integral is esponentially convergent at infinity when λ is in the upper half plane, we can invoke the Fubini-Tonelli theorem and **change the order of integrations**:

Analiticity of the Spectrum in the Upper Plane

$$\Rightarrow \oint_{\Gamma} P_{\mathrm{e}}\left(\lambda\right)\mathrm{d}\lambda = -\frac{\omega\mu_0}{4\pi E_0^{\mathrm{i}}}\int_0^{+\infty} J_{\mathrm{s}z}\left(x\right)\underbrace{\oint_{\Gamma} e^{+jk_0 x\lambda}\mathrm{d}\lambda}_{=0}\,\mathrm{d}x = 0$$

We can now make use of **Morera's theorem** to deduce that **the spectrum is analytic in the upper half plane**.

We will indicate this by setting it equal to an (unknown) "upper" half-plane function, namely:

$$\boxed{P_{\mathrm{e}}\left(\lambda\right) = U\left(\lambda\right)}$$

(note that this is only a temporary change in the name of the function P_{e})

Closing the Contour at Infinity

Note that, by closing the integration path by an infinite semicircular contour in the **upper half plane** (which is possible when $x < 0$), the analyticity of the spectrum and Cauchy's Theorem allow for concluding that **the second integral equation is satisfied**, i.e.,

$$\int_{-\infty}^{+\infty} U\left(\lambda\right) e^{-jk_0 x\lambda} \mathrm{d}\lambda = 0, \qquad x < 0$$

In order to solve the other integral equation, i.e.,

$$\int_{-\infty}^{+\infty} \frac{U\left(\lambda\right)}{\sqrt{1-\lambda^2}} e^{-jk_0 x\lambda} \mathrm{d}\lambda = -e^{jk_0 x \cos\phi_0}, \qquad x > 0$$

we will again close the integration path by an infinite semicircular contour, but this time in the **lower half plane**, since $x > 0$.

Closing the Contour at Infinity

To recover the right hand side of the latter integral equation by means of the **Residue Theorem** we write the integrand in the form

$$\frac{U(\lambda)}{\sqrt{1-\lambda^2}} = \frac{1}{2\pi j}\frac{L_1(\lambda)}{L_1(-\lambda_0)}\frac{1}{\lambda+\lambda_0} + L_2(\lambda)$$

where $L_{1,2}(\lambda)$ are unknown "lower" functions (i.e., functions analytic in the lower half λ-plane) and

$$\lambda_0 = \cos\phi_0$$ ('optical pole')

Despite the presence of three unknown functions, the above **functional equation** is sufficient to determine $U(\lambda)$...

Splitting Procedure

The standard approach is to decompose all functions into functions that are analytic as well free of zeros in either the upper or lower half-plane.

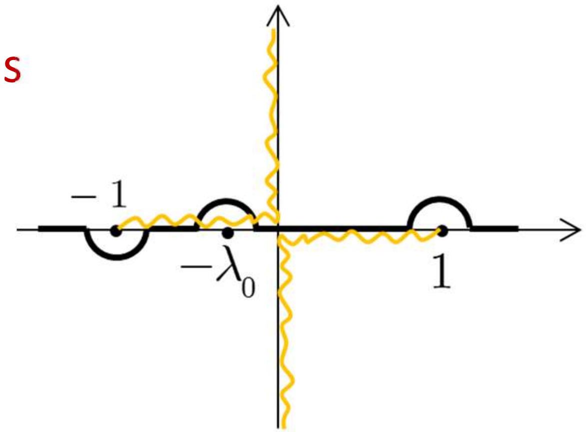
This procedure is commonly referred to as the **factorization** or **splitting** of a complex function as

$$F(\lambda) = F_{+}(\lambda) F_{-}(\lambda)$$

where $F_{+}(\lambda)$ is the "upper split" function and $F_{-}(\lambda)$ is the "lower split" function.

In this case, the function $\sqrt{1-\lambda^2}$ can be decomposed as

$$K(\lambda) = \sqrt{1-\lambda^2} = \underbrace{\sqrt{1-\lambda}}_{K_{+}(\lambda)} \underbrace{\sqrt{1+\lambda}}_{K_{-}(\lambda)}$$



complex λ-plane

Separating the Upper and Lower Functions

By inserting the split form into the functional equation we have:

$$\frac{U(\lambda)}{K_+(\lambda)} = \frac{1}{2\pi j}\frac{L_1(\lambda)}{L_1(-\lambda_0)}\frac{K_-(\lambda)}{\lambda+\lambda_0} + K_-(\lambda)L_2(\lambda)$$

In order to remove the optical pole (which lies in the lower half plane) from the right-hand side we may rewrite the previous equation as

$$\frac{U(\lambda)}{K_+(\lambda)} - \frac{1}{2\pi j}\frac{K_+(\lambda_0)}{\lambda+\lambda_0} = \frac{1}{2\pi j}\left[\frac{L_1(\lambda)}{L_1(-\lambda_0)}\underbrace{K_-(\lambda)}_{=K_+(-\lambda)} - K_+(\lambda_0)\right]\frac{1}{\lambda+\lambda_0} + K_-(\lambda)L_2(\lambda)$$

so that now the left hand side is an "upper function", whereas the right hand side is a "lower function"…

Behavior at Infinity

From the edge condition for the surface current density and **Watson's lemma** we have

$$J_z \underset{x\to 0}{\sim} x^{-1/2} \qquad \Rightarrow \qquad U(\lambda) = P_\mathrm{e}(\lambda) \propto \tilde{J}_z(k_0\lambda) \underset{\lambda\to\infty}{\sim} \lambda^{-1/2}$$

We deduce that the left hand side of the modified functional equation is **infinitesimal at infinity** (with order λ^{-1}):

$$\sim \lambda^{-1/2}$$

$$\frac{U(\lambda)}{K_+(\lambda)} - \frac{1}{2\pi j}\frac{K_+(\lambda_0)}{\lambda+\lambda_0} \sim \lambda^{-1}$$

$$\sim \lambda^{1/2}$$

The Crucial Step

- Since the left and right hand sides of the modified functional equations are upper and lower functions, respectively, they must both be **entire functions**.

- Furthermore, since they are infinitesimal at infinity, they are **bounded entire functions** and hence, by **Liouville Theorem**, they must be equal to a **constant**.

- Finally, since they are infinitesimal at infinity, such a constant must be **equal to zero**:

$$\frac{U(\lambda)}{K_{+}(\lambda)} - \frac{1}{2\pi j}\frac{K_{+}(\lambda_{0})}{\lambda + \lambda_{0}} = 0$$

The Solution for the Angular Spectrum

Hence

$$P_e(\lambda) = U(\lambda) = \frac{1}{2\pi j}\frac{K_+(\lambda)K_+(\lambda_0)}{\lambda+\lambda_0} = \frac{1}{2\pi j}\frac{\sqrt{1-\lambda}\sqrt{1-\lambda_0}}{\lambda+\lambda_0}$$

or

$$\boxed{P_e(\cos\alpha) = \frac{1}{2\pi j}\frac{\sqrt{1-\cos\alpha}\sqrt{1-\cos\phi_0}}{\cos\alpha+\cos\phi_0}}$$

The angular spectrum of the scattered field has thus been found in a **closed analytical form**.

References

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