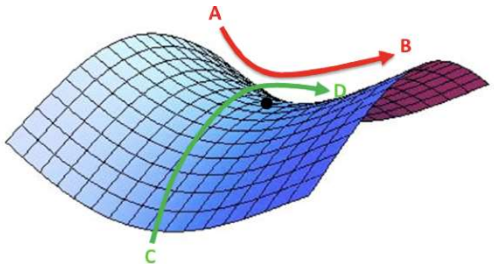
Ph.D. Course on

Analytical Techniques for Wave Phenomena



Lesson 7

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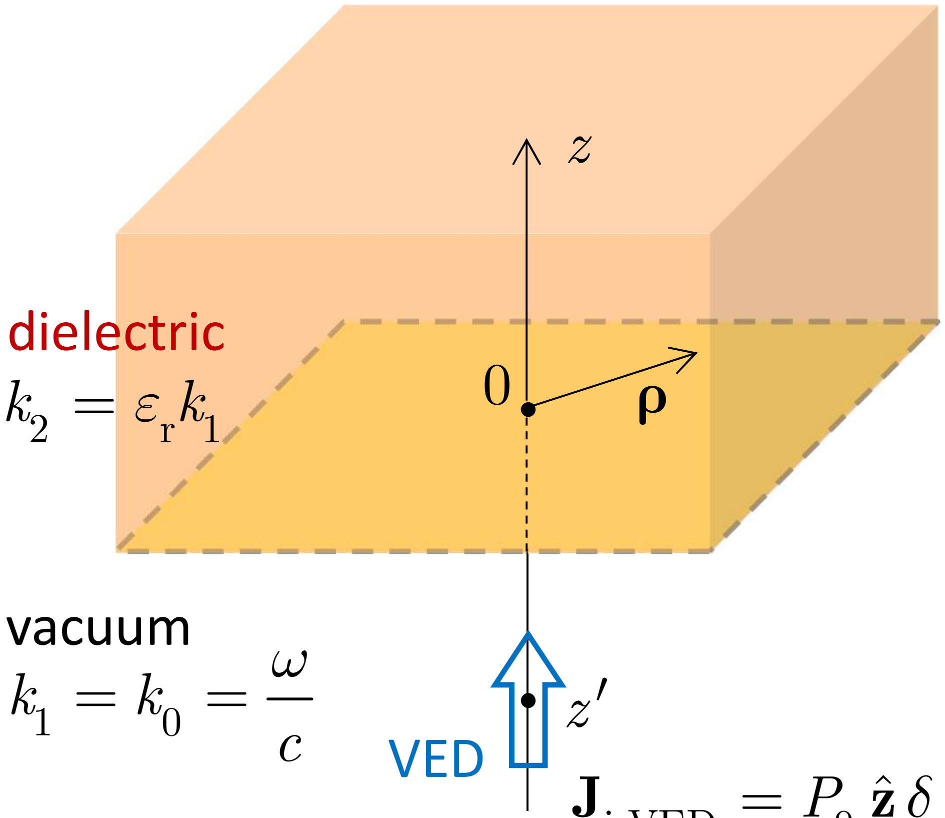
The Sommerfeld Half-Space Problem

Statement of the Problem

Sommerfeld problem:

Find the potential produced by a **Vertical Electric Dipole** (VED) placed in a vacuum in the presence of a (lossy) **dielectric half space**.



$$\mathbf{J}_{\mathrm{i,VED}} = P_0\, \hat{\mathbf{z}}\, \delta(\boldsymbol{\rho})\, \delta(z - z')$$

time-harmonic regime: $e^{j\omega t}$

radian frequency: ω

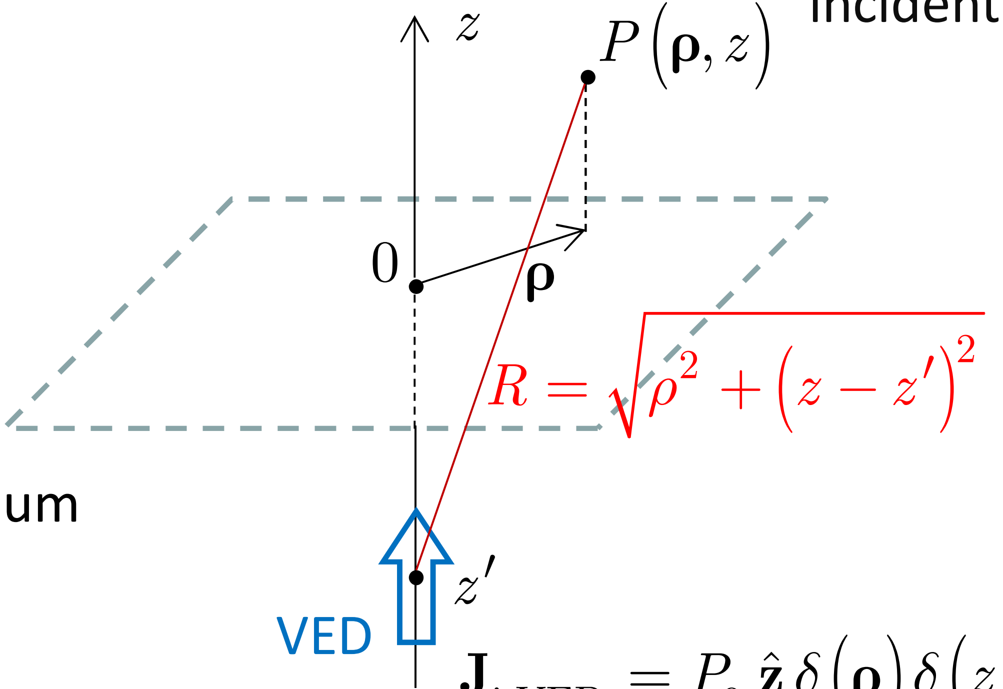
VED amplitude: $P_0 \; [\mathrm{A \cdot m}]$

The Incident Potential

The incident field is purely TM^z and hence it can be derived from a z-directed magnetic potential $\mathbf{A}^{\mathrm{i}} = \hat{\mathbf{z}}\, A_z^{\mathrm{i}}$

incident potential: $$A_z^{\mathrm{i}} = P_0 \frac{e^{-jk_1 R}}{4\pi R}$$

spherical wave



vacuum

$$\mathbf{J}_{\mathrm{i,VED}} = P_0\, \hat{\mathbf{z}}\, \delta(\boldsymbol{\rho})\, \delta(z - z')$$

Plane-Wave Spectrum of the Incident Potential

The interaction of a *spherical* wave with a *planar* interface **cannot be described analytically**.

Hence it is convenient to express the incident field as an *integral superposition of plane waves* (i.e., a **plane-wave spectrum**):

$$\frac{e^{-jk_0 R}}{4\pi R} = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2_{\mathbf{k}_\mathrm{t}}} \frac{e^{-jk_{z1}|z-z'|}}{2jk_{z1}} e^{-j\mathbf{k}_\mathrm{t}\cdot\boldsymbol{\rho}} \mathrm{d}\mathbf{k}_\mathrm{t}$$

Weyl identity

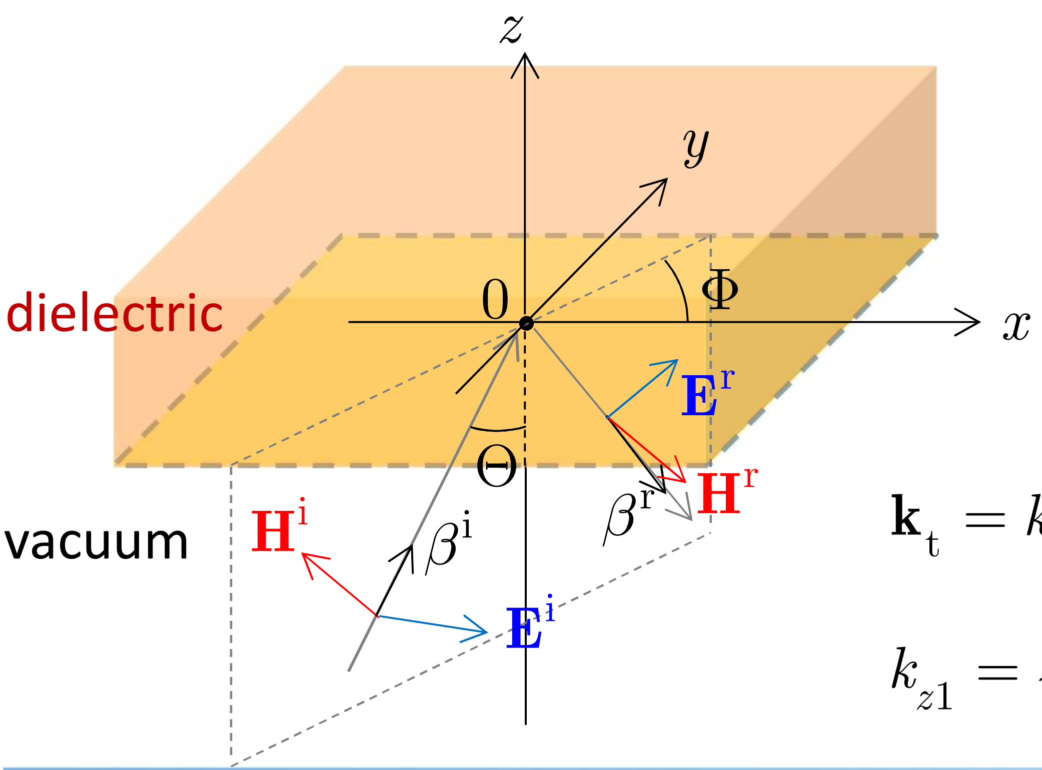
where $k_{z1} = \sqrt{k_1^2 - \mathbf{k}_\mathrm{t}\cdot\mathbf{k}_\mathrm{t}}$ with $\mathrm{Im}\{k_{z1}\} < 0$.

The latter condition ensures that the evanescent waves of the spectrum decay exponentially to zero as $z \to \pm\infty$.

Plane-Wave Reflection

The interaction between each plane-wave constituent of such a spectrum and the dielectric half space is readily described through the **TM reflection coefficient**:

$$\Gamma^{\mathrm{TM}} = \frac{Z_{02}^{\mathrm{TM}} - Z_{01}^{\mathrm{TM}}}{Z_{02}^{\mathrm{TM}} + Z_{01}^{\mathrm{TM}}} = \frac{k_{z2} / \left(\omega\varepsilon_2\right) - k_{z1} / \left(\omega\varepsilon_1\right)}{k_{z2} / \left(\omega\varepsilon_2\right) + k_{z1} / \left(\omega\varepsilon_1\right)} = \frac{k_{z2} - \varepsilon_{\mathrm{r}} k_{z1}}{k_{z2} + \varepsilon_{\mathrm{r}} k_{z1}}$$



- The reflected plane wave is also TM^z, i.e., the interface **does not introduce cross-polarization**.
- Thanks to rotational symmetry,

$$\Gamma^{\mathrm{TM}} = \Gamma^{\mathrm{TM}}\left(k_\rho = \left|\mathbf{k}_{\mathrm{t}}\right|\right)$$

$$\mathbf{k}_{\mathrm{t}} = k_1\left(\hat{\mathbf{x}}\sin\Theta\cos\Phi + \hat{\mathbf{y}}\sin\Theta\sin\Phi\right)$$

$$k_{z1} = \sqrt{k_1^2 - \mathbf{k}_{\mathrm{t}}\cdot\mathbf{k}_{\mathrm{t}}} = k_1\cos\Theta$$

The Reflected Potential

We thus have for the **reflected potential**:

$$A_z^{\mathrm{r}}\left(z<0\right)=P_0\,\frac{1}{\left(2\pi\right)^2}\iint\limits_{\mathbb{R}^2_{\mathbf{k}_{\mathrm{t}}}}\left(-\Gamma^{\mathrm{TM}}\right)\frac{e^{jk_{z1}\left(z+z'\right)}}{2jk_{z1}}\,e^{-j\mathbf{k}_{\mathrm{t}}\cdot\boldsymbol{\rho}}\,\mathrm{d}\mathbf{k}_{\mathrm{t}}$$

Note the **minus sign** in front of the reflection coefficient: this is due to the fact that the z-component of the magnetic potential (A_z) is proportional to the **current** in the transmission-line model of each plane wave:

$$\tilde{A}_z=L\left(z\right)T\left(\boldsymbol{\rho}\right)$$

$$T\left(\boldsymbol{\rho}\right)=e^{-j\mathbf{k}_{\mathrm{t}}\cdot\rho}$$

$$\tilde{\mathbf{H}}_{\mathrm{t}}=-L\nabla_{\mathrm{t}}T\times\hat{\mathbf{z}} \quad \text{current}$$

$$\tilde{\mathbf{E}}_{\mathrm{t}}=\frac{1}{j\omega\varepsilon}\frac{\mathrm{d}L}{\mathrm{d}z}\nabla_{\mathrm{t}}T \quad \text{voltage}$$

Now, Γ^{TM} is a *voltage* reflection coefficient; as is well known, to obtain the *current* reflection coefficient we just need to change its sign.

Potential at the Interface and Transmitted Potential

At the lower vacuum-dielectric interface ($z=0^-$) the **total potential** is:

$$A_z\left(z=0^-\right)=P_0\frac{1}{\left(2\pi\right)^2}\iint_{\mathbb{R}^2_{\mathbf{k}_t}}\left(1-\Gamma^{\mathrm{TM}}\right)\frac{e^{jk_{z1}z'}}{2jk_{z1}}e^{-j\mathbf{k}_t\cdot\boldsymbol{\rho}}\mathrm{d}\mathbf{k}_t$$

The continuity of the equivalent current across such an interface implies the continuity of A_z at $z=0$:

$$A_z\left(z=0^-\right)=A_z\left(z=0^+\right)=:A_z\left(z=0\right)$$

and hence for the **transmitted potential** we can write:

$$A_z\left(z>0\right)=P_0\frac{1}{\left(2\pi\right)^2}\iint_{\mathbb{R}^2_{\mathbf{k}_t}}\left(1-\Gamma^{\mathrm{TM}}\right)\frac{e^{jk_{z1}z'}e^{-jk_{z2}z}}{2jk_{z1}}e^{-j\mathbf{k}_t\cdot\boldsymbol{\rho}}\mathrm{d}\mathbf{k}_t$$

$$k_{z2}=\sqrt{k_2^2-\mathbf{k}_t\cdot\mathbf{k}_t}\qquad \mathrm{Im}\left\{k_{z2}\right\}<0$$

Total Potential

To summarize, we have:

$$A_z\left(\boldsymbol{\rho}, z\right) = P_0 \begin{cases} \dfrac{e^{-jk_1 R}}{4\pi R} + \dfrac{1}{\left(2\pi\right)^2} \iint\limits_{\mathbb{R}^2_{\mathbf{k}_\mathrm{t}}} \left(-\Gamma^{\mathrm{TM}}\right) \dfrac{e^{jk_{z1}\left(z+z'\right)}}{2jk_{z1}} e^{-j\mathbf{k}_\mathrm{t}\cdot\boldsymbol{\rho}} \mathrm{d}\mathbf{k}_\mathrm{t}, \ z \le 0 \\ \dfrac{1}{\left(2\pi\right)^2} \iint\limits_{\mathbb{R}^2_{\mathbf{k}_\mathrm{t}}} \left(1 - \Gamma^{\mathrm{TM}}\right) \dfrac{e^{jk_{z1}z'} e^{-jk_{z2}z}}{2jk_{z1}} e^{-j\mathbf{k}_\mathrm{t}\cdot\boldsymbol{\rho}} \mathrm{d}\mathbf{k}_\mathrm{t}, \ z \ge 0 \end{cases}$$

Exploiting the Rotational Symmetry

We now exploit the rotational symmetry of the configuration, by which all the previous expressions are of the kind:

$$I = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2_{\mathbf{k}_t}} \tilde{f}(k_\rho) e^{-j\mathbf{k}_t \cdot \boldsymbol{\rho}} \mathrm{d}\mathbf{k}_t$$

Performing the integration in **polar variables** $k_x = k_\rho \cos\Phi, \quad k_y = k_\rho \sin\Phi$:

$$(x = \rho\cos\phi, \quad y = \rho\sin\phi)$$

$$I = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{+\infty} \tilde{f}(k_\rho) e^{-j\mathbf{k}_t \cdot \boldsymbol{\rho}} k_\rho \mathrm{d}k_\rho \mathrm{d}\Phi$$

$$= \frac{1}{(2\pi)^2} \int_0^{+\infty} \tilde{f}(k_\rho) k_\rho \underbrace{\int_0^{2\pi} e^{-jk_\rho \rho \cos(\Phi - \phi)} \mathrm{d}\Phi}_{=2\pi J_0(k_\rho \rho)} \mathrm{d}k_\rho = \boxed{\frac{1}{2\pi} \int_0^{+\infty} \tilde{f}(k_\rho) J_0(k_\rho \rho) k_\rho \mathrm{d}k_\rho}$$

$$= I(\rho)$$

Sommerfeld integral $S_0\left\{\tilde{f}(k_\rho)\right\}$

Total Potential (Axial Transmission Representation)

As a consequence, the potential does not depend on ϕ; it can be expressed in terms of Sommerfeld integrals as

$$A_z\left(\rho, z\right) = P_0 \begin{cases} \dfrac{e^{-jk_1 R}}{4\pi R} + \dfrac{1}{2\pi}\displaystyle\int_0^{+\infty}\left(-\Gamma^{\mathrm{TM}}\right)\dfrac{e^{jk_{z1}\left(z+z'\right)}}{2jk_{z1}} J_0\left(k_\rho \rho\right) k_\rho \mathrm{d}k_\rho, \; z \le 0 \\ \dfrac{1}{2\pi}\displaystyle\int_0^{+\infty}\left(1-\Gamma^{\mathrm{TM}}\right)\dfrac{e^{jk_{z1}z'}e^{-jk_{z2}z}}{2jk_{z1}} J_0\left(k_\rho \rho\right) k_\rho \mathrm{d}k_\rho, \; z \ge 0 \end{cases}$$

Since in each of the above integrals the integrand is an elementary wave propagating along the axial z-direction, this is also called an Axial Transmission Representation of the potential.

Singularities in the k_ρ-Plane

Since the Bessel function J_0 is an *entire function* of its argument, the singularities of the integrand arise from:

- The zeros of the denominator of the reflection coefficient Γ^{TM} (these are **poles**):

$$\Gamma^{\mathrm{TM}} = \frac{k_{z2} - \varepsilon_{\mathrm{r}} k_{z1}}{k_{z2} + \varepsilon_{\mathrm{r}} k_{z1}}$$

- The square-root functions defining the vertical wavenumbers k_{z1} and k_{z2} (these introduce **branch points** and make the integrand **multivalued**):

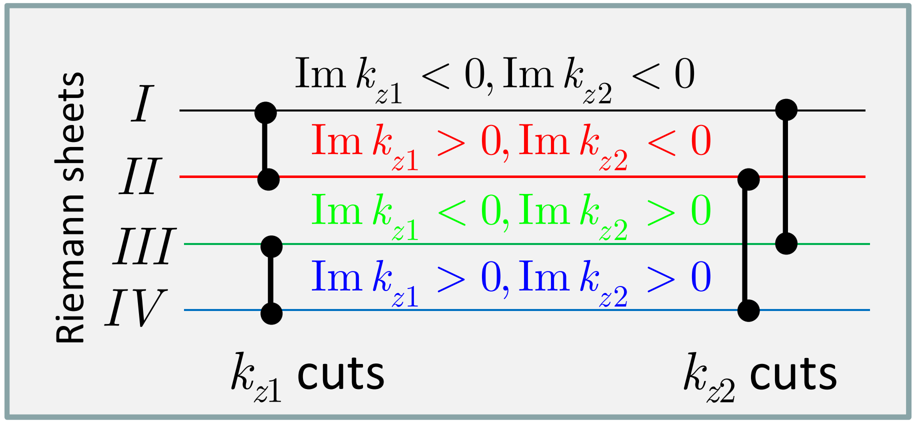
$$k_{z1} = \sqrt{k_1^2 - k_\rho^2} \qquad k_{z2} = \sqrt{k_2^2 - k_\rho^2} = \sqrt{\varepsilon_{\mathrm{r}} k_1^2 - k_\rho^2}$$

Branch Points and Branch Cuts in the k_ρ-Plane

The branch points (BPs) introduced by the square-root functions are located at

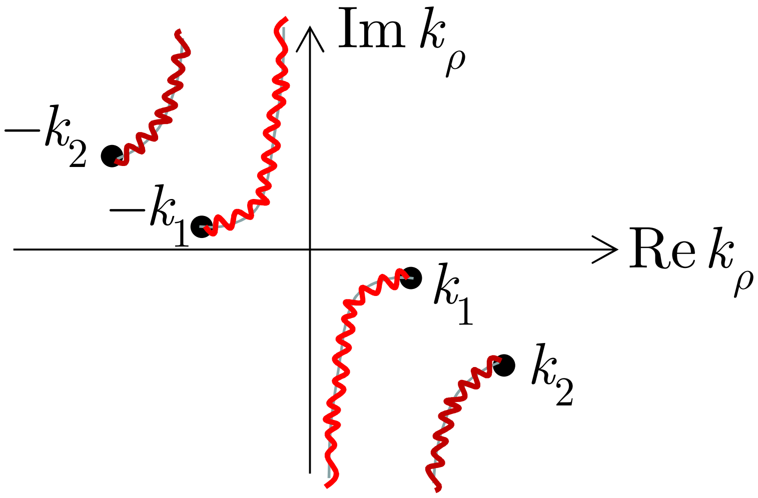
$$k_\rho = \pm k_1 \qquad k_\rho = \pm k_2 = \pm k_1\sqrt{\varepsilon_{\mathrm{r}}}$$

To make the integrand single-valued, **two pairs of branch cuts** are introduced, which define a **four-sheeted Riemann surface**:

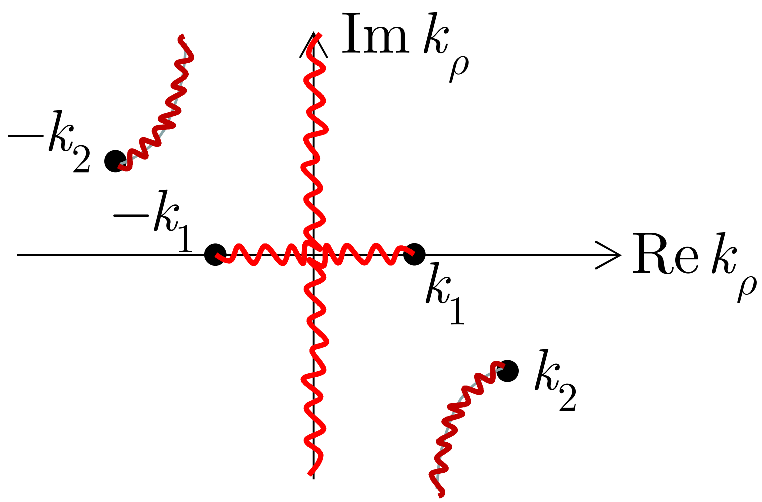


Sommerfeld Branch Cuts

The shape of the cuts is quite arbitrary, but the hyperbolic cuts defined by the condition $\mathrm{Im}\, k_{z1,z2} = 0$ (called the fundamental or **Sommerfeld branch cuts**) are especially convenient, since they allow to **easily enforce the radiation condition at infinity** by keeping the integration path on the '*proper*' (or '*top*') sheet I, where $\mathrm{Im}\, k_{z1} < 0, \mathrm{Im}\, k_{z2} < 0$.



medium 1 lossy
medium 2 lossy



medium 1 lossless
medium 2 lossy

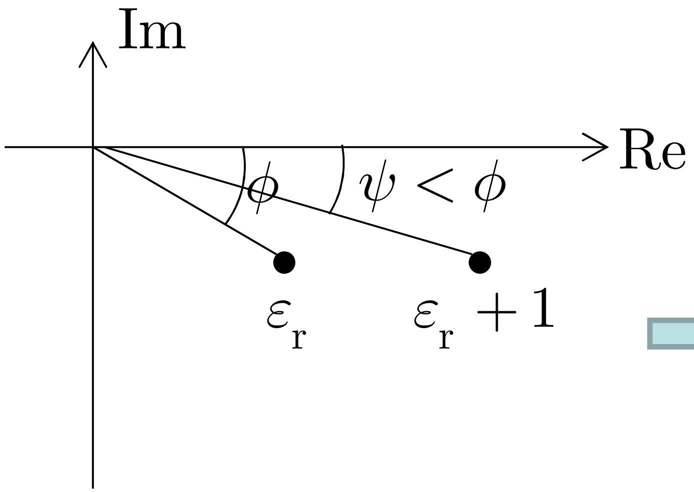
The Sommerfeld Poles in the k_ρ-Plane

By equating to zero the denominator of Γ^{TM} we have the **dispersion equation**:

$$\varepsilon_{\mathrm{r}}\sqrt{k_1^2 - k_\rho^2} = -\sqrt{\varepsilon_{\mathrm{r}} k_1^2 - k_\rho^2}$$

whence squaring we find $k_\rho = \pm k_{\mathrm{p}} = \pm k_1 \sqrt{\dfrac{\varepsilon_{\mathrm{r}}}{\varepsilon_{\mathrm{r}} + 1}}$ **Sommerfeld poles**

The quadrant of the complex plane where k_{p} resides can be inferred from:



$$\Rightarrow \quad k_1 \sqrt{\frac{|\varepsilon_{\mathrm{r}}|}{|\varepsilon_{\mathrm{r}} + 1|}} e^{-j(\phi - \psi)/2} \in \mathrm{IV\ quadrant}$$

However, squaring obliterates any distinction of the square-root branches...

The Sommerfeld Poles in the k_ρ-Plane

Substituting $k_\rho = \pm k_\mathrm{p}$ in the dispersion equation: $\varepsilon_\mathrm{r}\sqrt{\dfrac{k_1^2}{\varepsilon_\mathrm{r}+1}} = -\sqrt{\dfrac{k_1^2\varepsilon_\mathrm{r}^2}{\varepsilon_\mathrm{r}+1}}$

We assume that the loss angle of medium 2 (dielectric) exceeds that of medium 1 (vacuum, air), so that we may write $k_1^2 = \left|k_1^2\right| e^{-j\delta}, \quad 0 \le \delta < \psi$

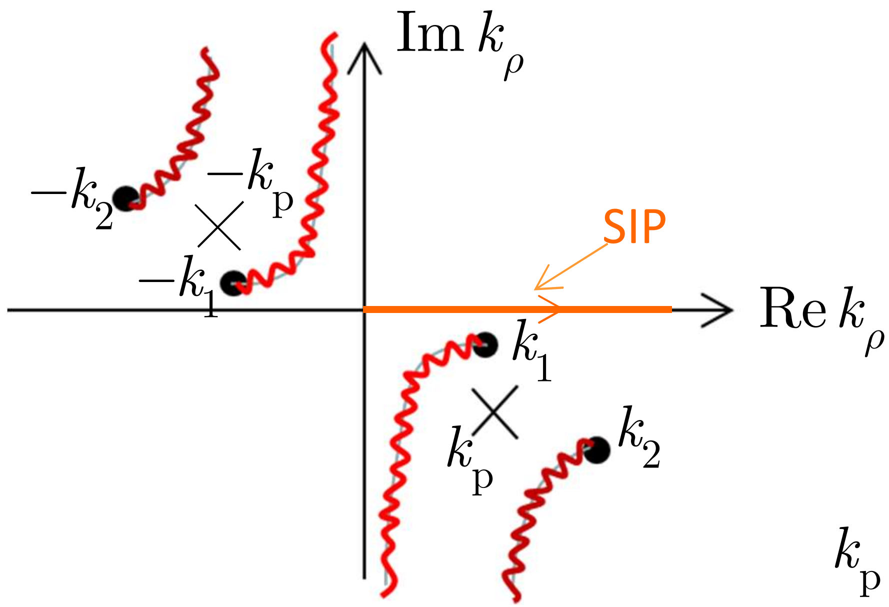
$$\Rightarrow \quad \sqrt{\frac{k_1^2}{\varepsilon_\mathrm{r}+1}} = -\sqrt{\frac{\left|k_1^2\right|}{\left|\varepsilon_\mathrm{r}+1\right|}}\, e^{j(\psi-\delta)/2} \qquad \sqrt{\frac{k_1^2\varepsilon_\mathrm{r}^2}{\varepsilon_\mathrm{r}+1}} = \left|\varepsilon_\mathrm{r}\right|\sqrt{\frac{\left|k_1^2\right|}{\left|\varepsilon_\mathrm{r}+1\right|}}\, e^{-j[\phi-(\psi-\delta)/2]}$$

The imaginary parts of the square roots have been chosen *negative* and it can be checked that this makes the LHS and RHS of the dispersion equation match, so that **the Sommerfeld poles lie on the top sheet (sheet I)**.

Remark: The poles also exists on sheet IV, whereas on sheets II and III the above square roots satisfy the dispersion equation with the sign of the RHS reversed, i.e., they are zeros of the *numerator* of Γ^{TM} on these sheets (*Brewster zeros*).

Sommerfeld Integration Path in the k_ρ-Plane

The **Sommerfeld integration path (SIP)** is the *positive real axis* on the top Riemann sheet:



$$k_p = k_1 \sqrt{\frac{\varepsilon_r}{\varepsilon_r + 1}} \xrightarrow[|\varepsilon_r| \to +\infty]{} k_1$$

Note that, for high media contrast, the Sommerfeld poles may be arbitrarily close to the branch points $\pm k_1$. This causes a spike in the integrand. In addition, the integrand oscillates rapidly if $k_1 \rho \gg 1$ and $z \ll \rho$.

A Typical Sommerfeld Integrand

This is the magnitude (in logarithmic scale) of a typical Sommerfeld integrand along the SIP for a VED above a lossy ground:

Parameters:

$z' = -5\ \text{m}$

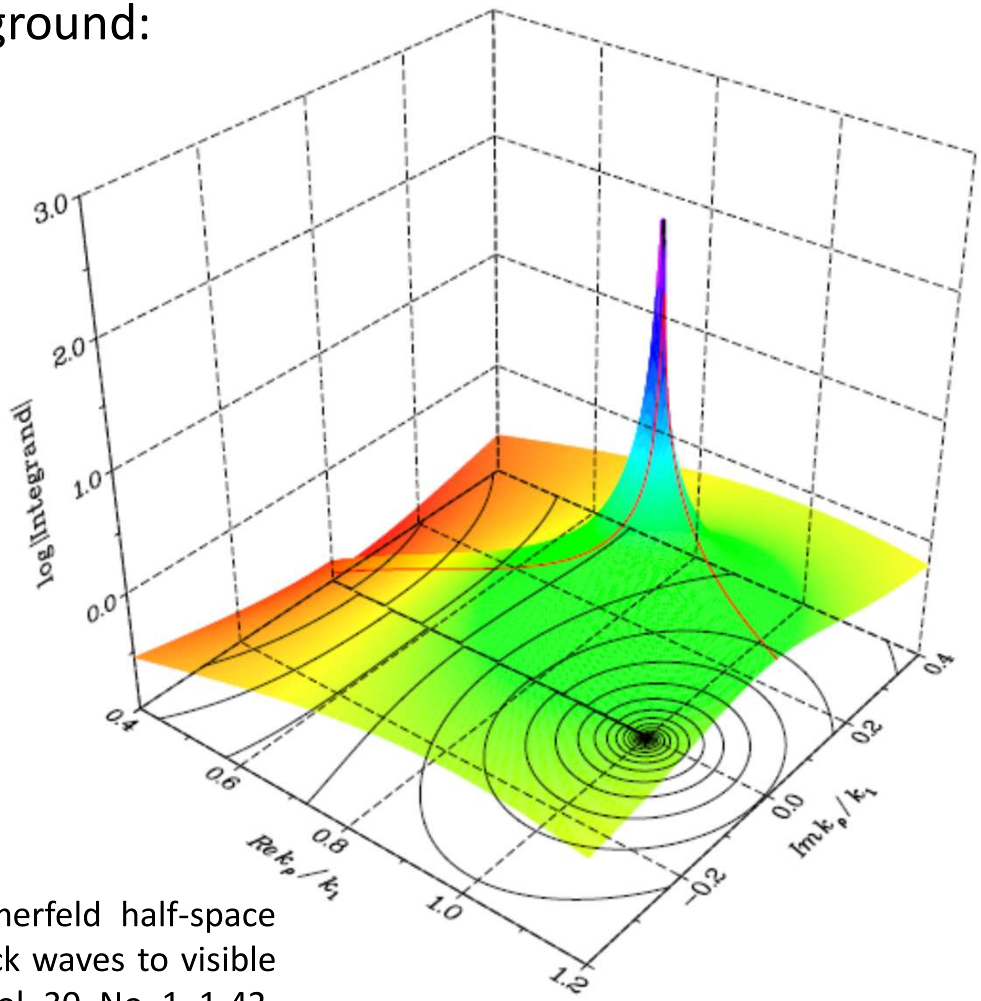
$z = 0\ \text{m}$

$\rho = 50\ \text{m}$

$\sigma = 10\,\text{mS/m}$

$f = 1\,\text{MHz}$

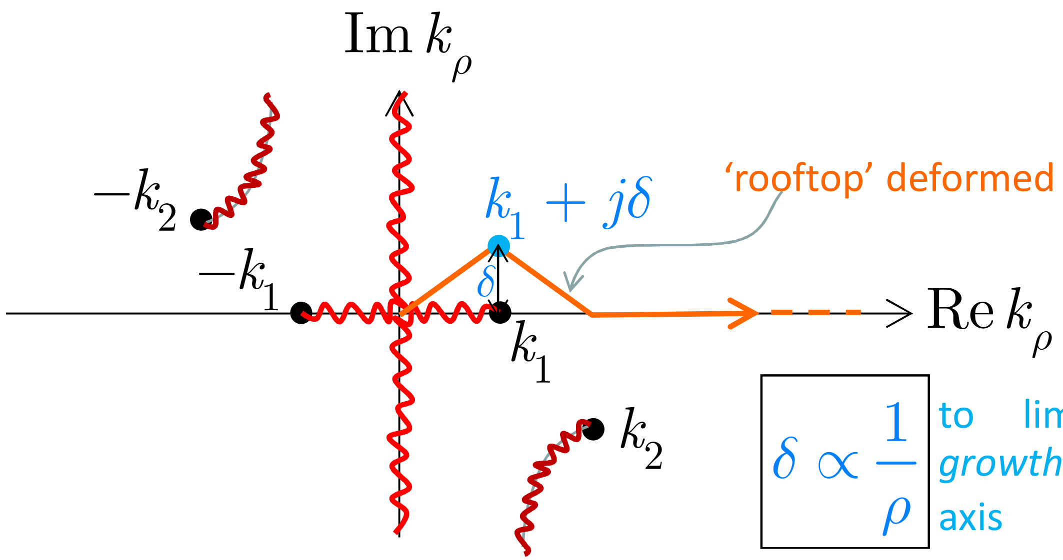
$\rightarrow \varepsilon_{\text{r}} = 10 - j180$



[From K. A. Michalski and J. R. Mosig, "The Sommerfeld half-space problem revisited: from radio frequencies and Zenneck waves to visible light and Fano modes," *J. Electromagn. Waves Appl.*, Vol. 30, No. 1, 1-42, 2016]

Avoiding the Spike: Detour in the 1st Quadrant

A simple remedy to the numerical problems produced by the presence of the Sommerfeld pole is a **detour** in the first quadrant of the complex k_ρ-plane, which **skirts the singularities** and then **rejoins the real axis**:



$$\delta \propto \frac{1}{\rho}$$

to limit the *exponential growth* of $J_0(k_\rho \rho)$ off the real axis

Standard quadratures can be used on the rooftop. The real-axis tail can be dealt with by using one of the effective existing approaches to the numerical evaluation of real Sommerfeld integrals.

Extended Sommerfeld Path in the k_ρ-Plane

Alternative representations for the potential, more amenable to *numerical evaluation* and capable of providing *physical insight* into the involved wave penomena, can be obtained by suitable **deformations** of the integration path.

To this aim, we first **extend** the SIP into a path along the **whole real axis**, using the relations:

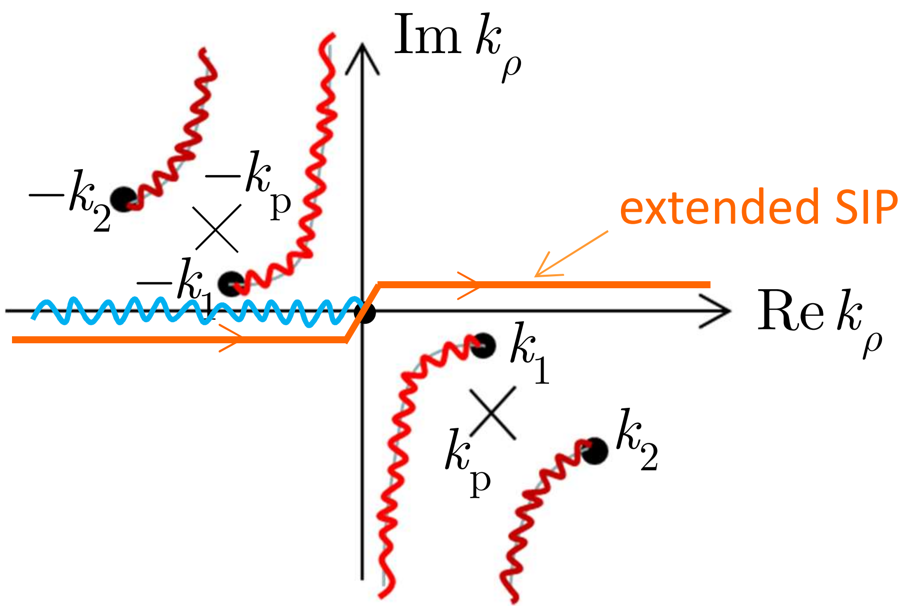
$$J_0\left(z\right) = \frac{1}{2}\left[H_0^{(1)}\left(z\right) + H_0^{(2)}\left(z\right)\right] \qquad H_0^{(1)}\left(ze^{j\pi}\right) = -H_0^{(2)}\left(z\right)$$

whence

$$A_z\left(\rho,z\right) = P_0 \begin{cases} \dfrac{e^{-jk_1R}}{4\pi R} + \dfrac{1}{4\pi}\displaystyle\int_{\infty e^{-j\pi}}^{\infty}\left(-\Gamma^{\mathrm{TM}}\right)\frac{e^{jk_{z1}\left(z+z'\right)}}{2jk_{z1}}H_0^{(2)}\left(k_\rho\rho\right)k_\rho \mathrm{d}k_\rho, \; z \le 0 \\ \dfrac{1}{4\pi}\displaystyle\int_{\infty e^{-j\pi}}^{\infty}\left(1-\Gamma^{\mathrm{TM}}\right)\frac{e^{jk_{z1}z'}e^{-jk_{z2}z}}{2jk_{z2}}H_0^{(2)}\left(k_\rho\rho\right)k_\rho \mathrm{d}k_\rho, \; z \ge 0 \end{cases}$$

Extended Sommerfeld Path in the k_ρ-Plane

Note that the Hankel function introduces an additional logarithmic branch point at $k_\rho = 0$ (devoid of any physical meaning), with its associated branch cut along the *negative real axis*. The integration path extends on the *lower rim* of this cut:



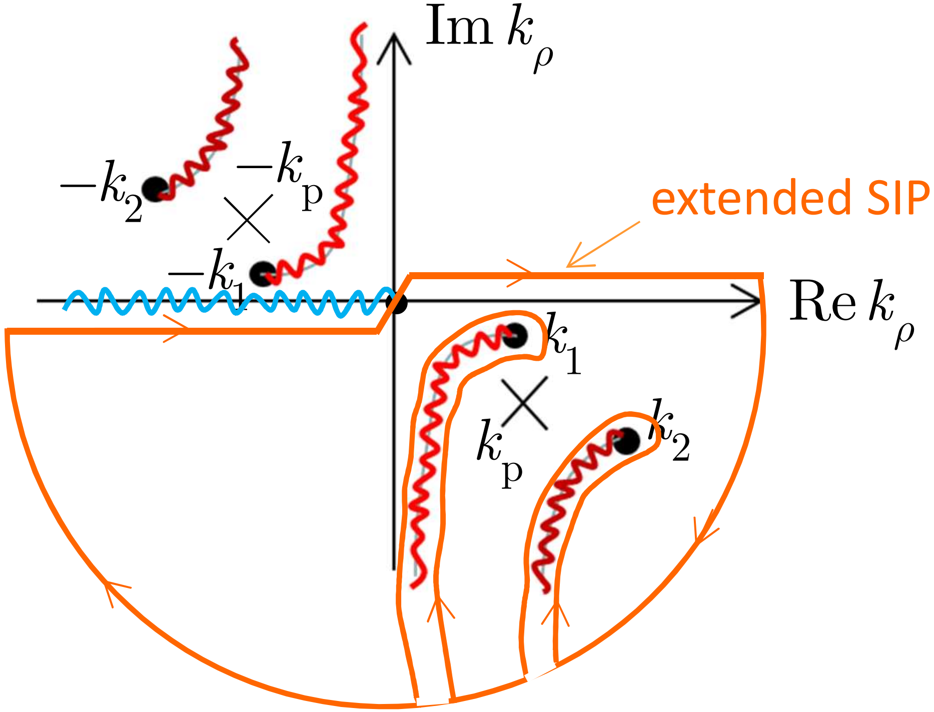
Note also that, since $H_0^{(2)}(z)$ is singular at $z = 0$, the representations of the potential using the extended SIP or integration paths derived thereof cannot be used for calculating the potential at $\rho = 0$.

Path Deformation in the Lower Half k_ρ-Plane

In view of the asymptotic behavior of the Hankel function for large arguments:

$$H_0^{(2)}(z) \sim \sqrt{\frac{2j}{\pi z}} e^{-jz}, \; |z| \to +\infty, \; -2\pi + \delta \le \arg z \le \pi - \delta$$

we may close the extended SIP in the lower half plane through a **semicircular arc at infinity**, *wrapping around the branch cuts*; such arc contributes nothing to the integral:

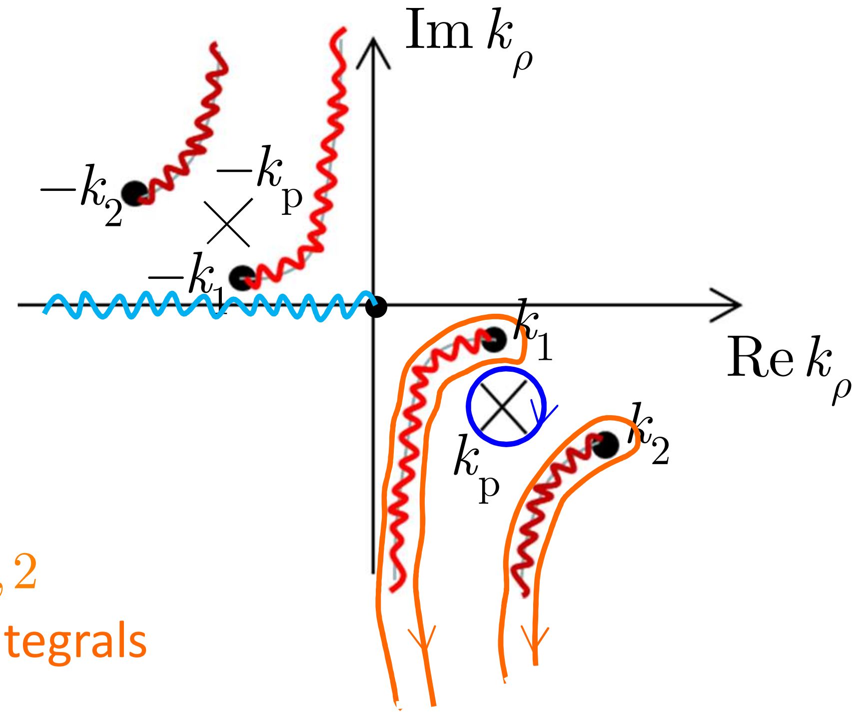


Radial Transmission Representation

Using the **Residue Theorem**, we obtain

$$A_z = \int_{\infty e^{-j\pi}}^{\infty} \tilde{f}\left(k_\rho\right) H_0^{(2)}\left(k_\rho \rho\right) k_\rho \mathrm{d}k_\rho =$$

$$= Q_1\left\{\tilde{f}\right\} + Q_2\left\{\tilde{f}\right\} - 2\pi j R_\mathrm{p} H_0^{(2)}\left(k_\mathrm{p}\rho\right) k_\mathrm{p}$$



where

$$Q_i\left\{\tilde{f}\right\} = \int_{\mathrm{BC}_i} \tilde{f}\left(k_\rho\right) H_0^{(2)}\left(k_\rho \rho\right) k_\rho \mathrm{d}k_\rho, \quad i = 1,2$$

branch-cut integrals

$$R_\mathrm{p} = \mathrm{Res}\left[\tilde{f}\left(k_\rho\right); k_\rho = k_\mathrm{p}\right]$$

residue contribution

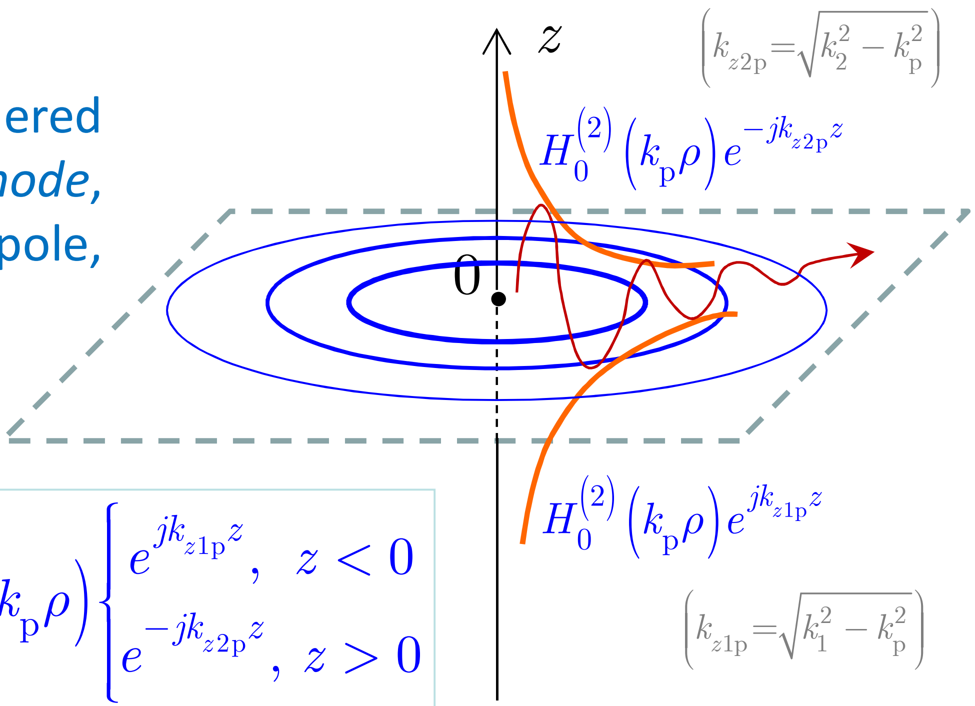
Since each of the above terms is expressed in terms of elementary waves propagating also the radial ρ-direction, this is called a Radial Transmission Representation of the potential.

Spectral Representation: the Zenneck Wave

The Radial Transmission Representation is also called a **Spectral Representation**, since it exhibits the potential as a sum of **modes** supported by the air-dielectric structure.

The BC integrals provide a **continuous spectrum** of modes, whereas the residue contributions of the pole singularity provides the **discrete spectrum**.

The discrete spectrum of the considered structure consists of a *single mode*, corresponding to the Sommerfeld pole, known as the **Zenneck surface wave**:



$$A_{zp}\left(\rho, z\right) = \mathrm{Res}\left\{\Gamma^{\mathrm{TM}}; k_p\right\} \frac{k_p e^{jk_{z1p} z'}}{4k_{z1p}} H_0^{(2)}\left(k_p \rho\right) \begin{cases} e^{jk_{z1p} z}, & z < 0 \\ e^{-jk_{z2p} z}, & z > 0 \end{cases}$$

Zenneck Waves and Brewster Angles

The wavenumber $k_\rho = k_{\mathrm{p}}$ is also **a zero of the reflection coefficient**. As such, it defines a (complex) **Brewster angle** θ_{B}:

$$k_{\mathrm{p}} = \mathrm{k}_1 \sin\theta_{\mathrm{B}} \;\; \rightarrow \theta_{\mathrm{B}} = \arcsin\sqrt{\frac{\varepsilon_{\mathrm{r}}}{\varepsilon_{\mathrm{r}}+1}} = \arctan\sqrt{\varepsilon_{\mathrm{r}}}$$

Only the incident and transmitted waves exist at this angle, and both must decay in magnitude away from the interface, in order to constitute a surface wave. However, the incident wave propagates *toward* the interface, so it must be $\mathrm{Im}k_{z1} > 0$, whereas for the transmitted wave $\mathrm{Im}k_{z2} < 0$. So k_{p} is a zero of the reflection coefficient **on Riemann sheet II**.

Remark: exchanging the role of the two half spaces, one can show that the (improper) Zenneck wave associated with the pole $k_\rho = k_{\mathrm{p}}$ on sheet IV corresponds to a zero of the reflection coefficient on sheet III, i.e., to the (complex) Brewster angle for incidence from the dielectric.

Spectral Representation: the Branch-Cut Integrals

The BC integrals can be written using $k_{z1,2}$ as variables of integration:

$$Q_i\left\{\tilde{f}\right\} = \int_0^{+\infty} \left[\!\left[\tilde{f}\left(k_\rho\right) \right]\!\right] H_0^{(2)}\left(k_\rho \rho\right) \Big|_{k_\rho = \sqrt{k_i^2 - k_{zi}^2}} k_{zi} \mathrm{d}k_{zi}, \qquad i = 1, 2$$

where $\left[\!\left[\tilde{f}\left(k_\rho\right) \right]\!\right] = \tilde{f}^+\left(k_\rho\right) - \tilde{f}^-\left(k_\rho\right)$ is the **folded** spectral integrand function

(i.e., the difference between the values of the original integrand calculated on the two rims of the relevant BC; the plus (minus) superscript indicates that every occurrence of k_{zi} is directly replaced by the variable of integration with a plus (minus) sign).

Note that the closeness of the Sommerfeld poles to the branch points and the rapid oscillations of the integrand for large $k_1\rho$ **still adversely affect** the numerical efficiency of the Radial Transmission Representation.

Nonspectral Representation

The numerical drawbacks of both the Axial and the Radial Transmission Representations can be overcome through a **further path deformation to the steepest-descent path (SDP)**, which leads to the so-called Nonspectral Representation.

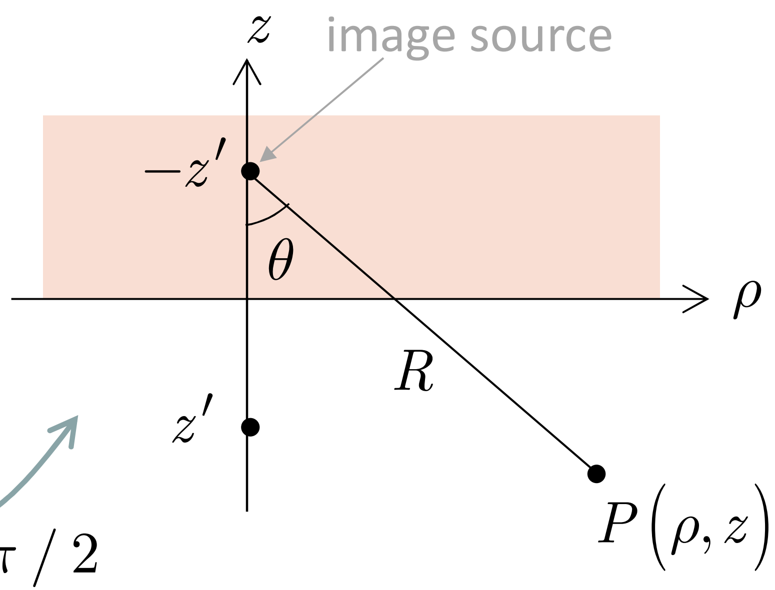
Let us consider for definiteness the reflected potential:

$$\frac{1}{4\pi}\int_{\infty e^{-j\pi}}^{\infty}\left(-\Gamma^{\mathrm{TM}}\right)\frac{e^{jk_{z1}\left(z+z'\right)}}{2jk_{z1}}H_0^{(2)}\left(k_\rho \rho\right)k_\rho \mathrm{d}k_\rho$$

with

$$\left.\begin{aligned}\rho &= R\sin\theta\\ -\left(z+z'\right) &= R\cos\theta\end{aligned}\right\}$$

$$0 < \theta < \pi / 2$$



We wish to evaluate this asymptotically for large observation distances: $k_1 R \gg 1$

Preaparation to Canonical Form for SD Evaluation

To this aim, let us single out the first-order asymptotic expansion of the Hankel function for large arguments:

$$\frac{1}{4\pi}\sqrt{\frac{2j}{\pi R\sin\theta}}\int_{\infty e^{-j\pi}}^{\infty}\left(-\Gamma^{\mathrm{TM}}\right)\frac{e^{-jk_{z1}R\cos\theta}}{2jk_{z1}}\frac{e^{-jk_\rho R\sin\theta}}{\sqrt{k_\rho}}\mathcal{H}_0^{(2)}\left(k_\rho R\sin\theta\right)k_\rho \mathrm{d}k_\rho$$

having introduced the normalized Hankel function

$$\mathcal{H}_0^{(2)}\left(\xi\right)=\sqrt{\frac{\pi\zeta}{2j}}e^{j\xi}H_0^{(2)}\left(\xi\right)\sim 1,\quad \xi\rightarrow\infty$$

This is in the **canonical form**: $I\left(k_1 R\right)=\int_C f\left(k_\rho\right)e^{k_1 R\,\phi\left(k_\rho\right)}\mathrm{d}k_\rho$

$$f\left(k_\rho\right)=\frac{1}{4\pi}\sqrt{\frac{2j}{\pi R\sin\theta}}\left(-\Gamma^{\mathrm{TM}}\right)\frac{k_\rho^{1/2}}{2jk_{z1}}\mathcal{H}_0^{(2)}\left(k_\rho R\sin\theta\right)\qquad \phi\left(k_\rho\right)=-\frac{j}{k_1}\left[k_{z1}\cos\theta+k_\rho\sin\theta\right]$$

Saddle Points and Geometrical Optics

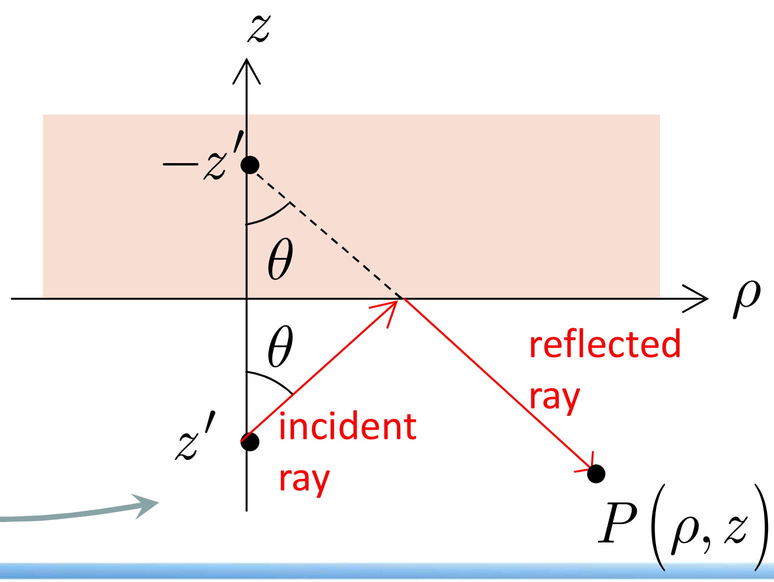
Let us determine the **Saddle Points** (SPs):

$$\phi'\left(k_\rho\right) = -\frac{j}{k_1}\left[-\frac{k_\rho}{k_{z1}}\cos\theta + \sin\theta\right] = 0 \quad \Rightarrow \quad \boxed{k_{\mathrm{SP}} = \pm k_1 \sin\theta}$$

First-order SP

$$\phi''\left(k_{\mathrm{SP}}\right) = \frac{j}{k_1^2 \cos^2\theta} = \left|\phi''\left(k_{\mathrm{SP}}\right)\right| e^{j\alpha} \neq 0 \quad \left(\alpha = \frac{\pi}{2}\right)$$

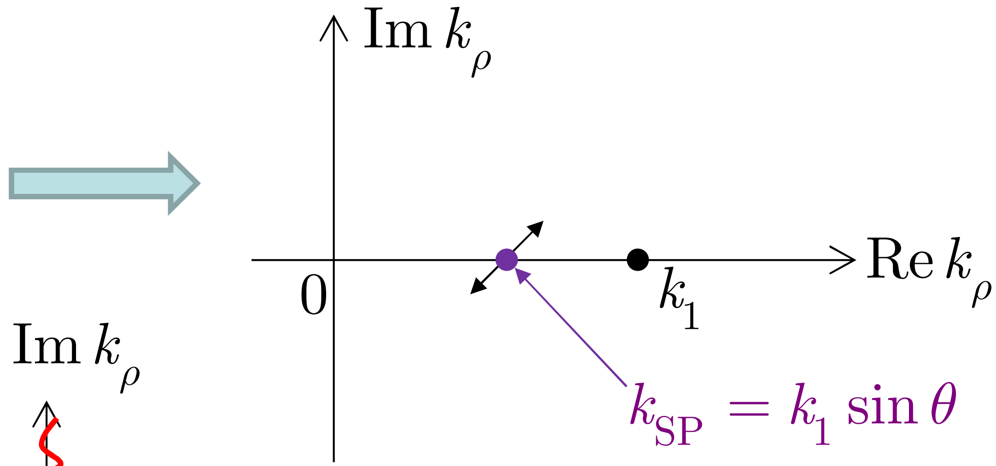
The SP determines the **Geometrical Optics** (GO) approximation to the reflected potential, as it selects the plane-wave spectral contribution that gives rise to the reflected ray



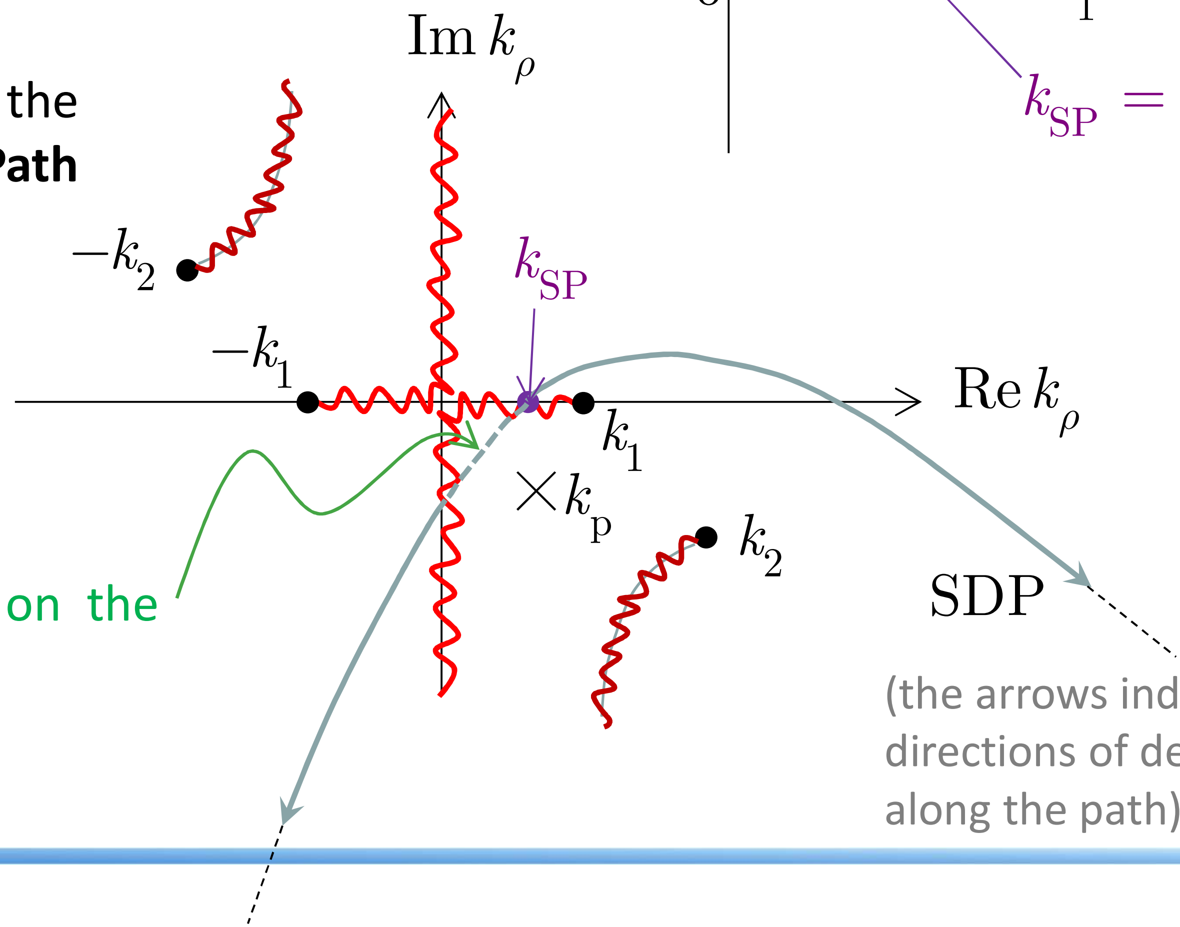
Steepest-Descent Path

steepest descent directions:

$$-\frac{\alpha}{2}+(2m+1)\frac{\pi}{2}=\frac{\pi}{4},\frac{5\pi}{4}$$



This is the shape of the **Steepest-Descent Path** (SDP) :



The SDP partly lies on the *improper* sheet II…

(the arrows indicate the directions of descent along the path)

SP Contribution: the GO Reflected Potential

The function $f(k_\rho)$ is regular in the vicinity of the SP, with

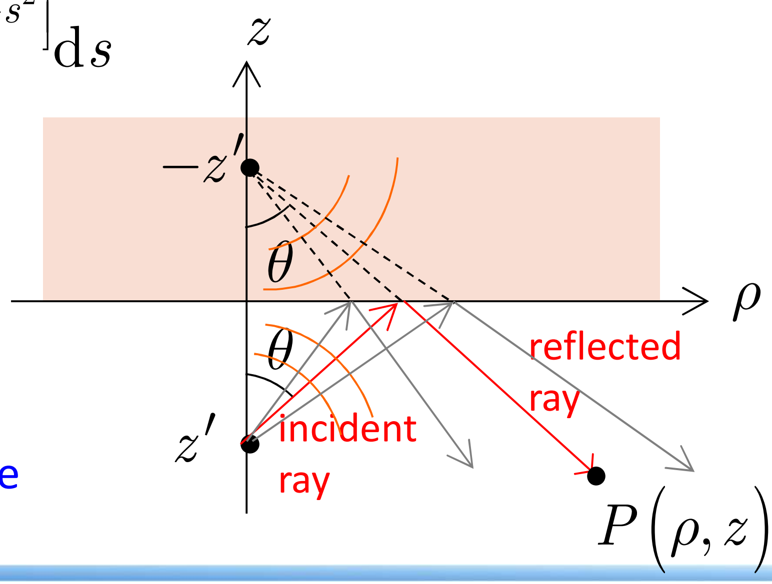
$$f\left(k_{\mathrm{SP}}\right)=\frac{1}{4\pi}\sqrt{\frac{2j}{\pi R}}\left(-\Gamma^{\mathrm{TM}}\left(k_{\mathrm{SP}}\right)\right)\frac{1}{2j}\frac{1}{k_1^{1/2}\cos\theta}\mathcal{H}_0^{(2)}\left(k_1 R\sin^2\theta\right)$$

By letting $\phi\left(k_\rho\right)=\phi\left(k_{\mathrm{SP}}\right)-s^2 \;\rightarrow\; \left(\frac{\mathrm{d}k_\rho}{\mathrm{d}s}\right)_{s=0}=\sqrt{\frac{-2}{\phi''\left(k_{\mathrm{SP}}\right)}}=k_1\cos\theta\sqrt{2j}$

$$I\left(k_1 R\right)\sim f\left(k_{\mathrm{SP}}\right)\left(\frac{\mathrm{d}k_\rho}{\mathrm{d}s}\right)_{s=0}\int_{-\infty}^{+\infty} e^{k_1 R\left[\phi\left(k_{\mathrm{SP}}\right)-s^2\right]}\mathrm{d}s$$

$$=\ldots$$

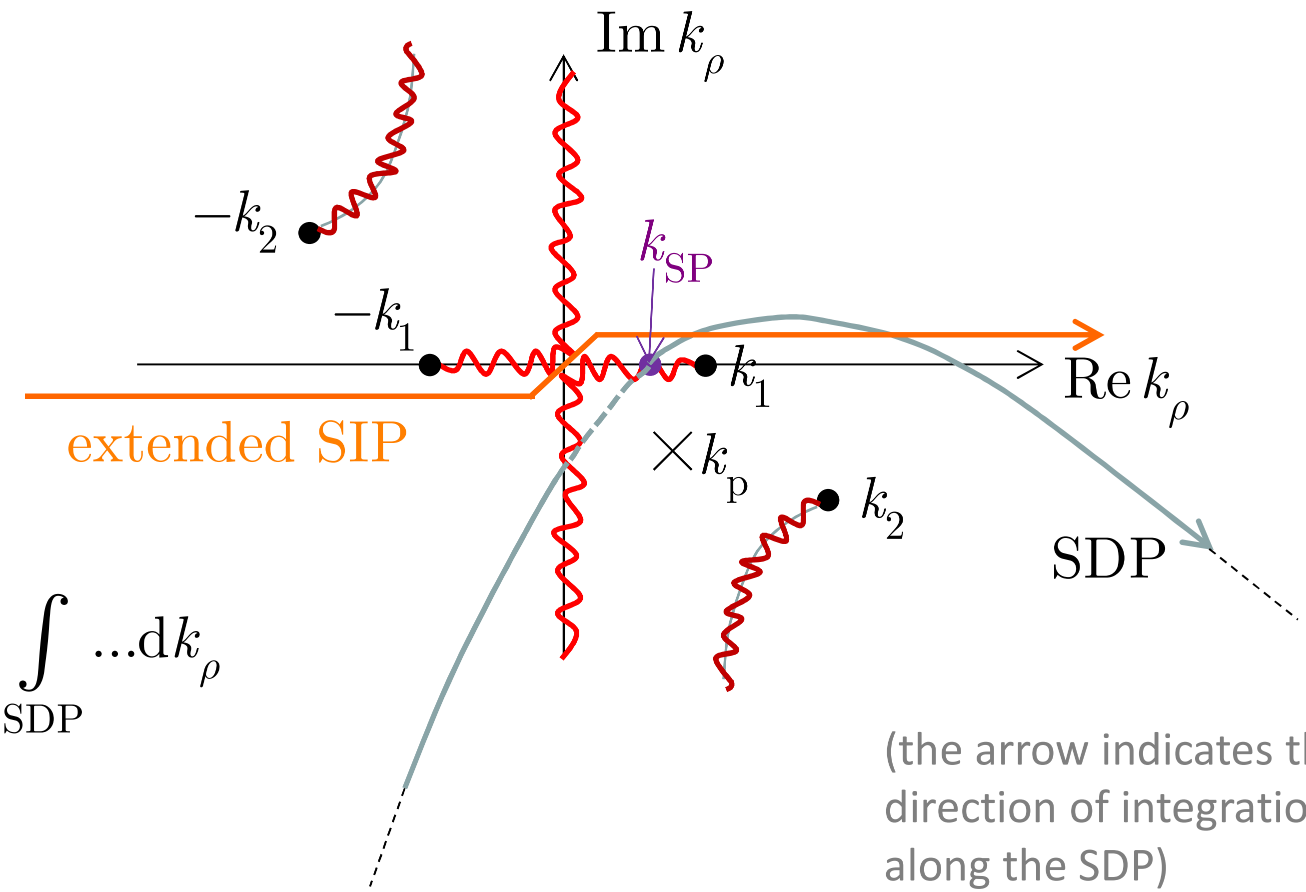
$$\sim -\Gamma^{\mathrm{TM}}\left(k_{\mathrm{SP}}\right)\frac{e^{-jk_1 R}}{4\pi R}$$



spherical wave originating from the image source and weighted by the reflection coefficient

Steepest-Descent Path as a Function of the Angle θ

Depending on the angle θ, the integral along the SDP *may or may not* be the only contribution to the reflected potential…

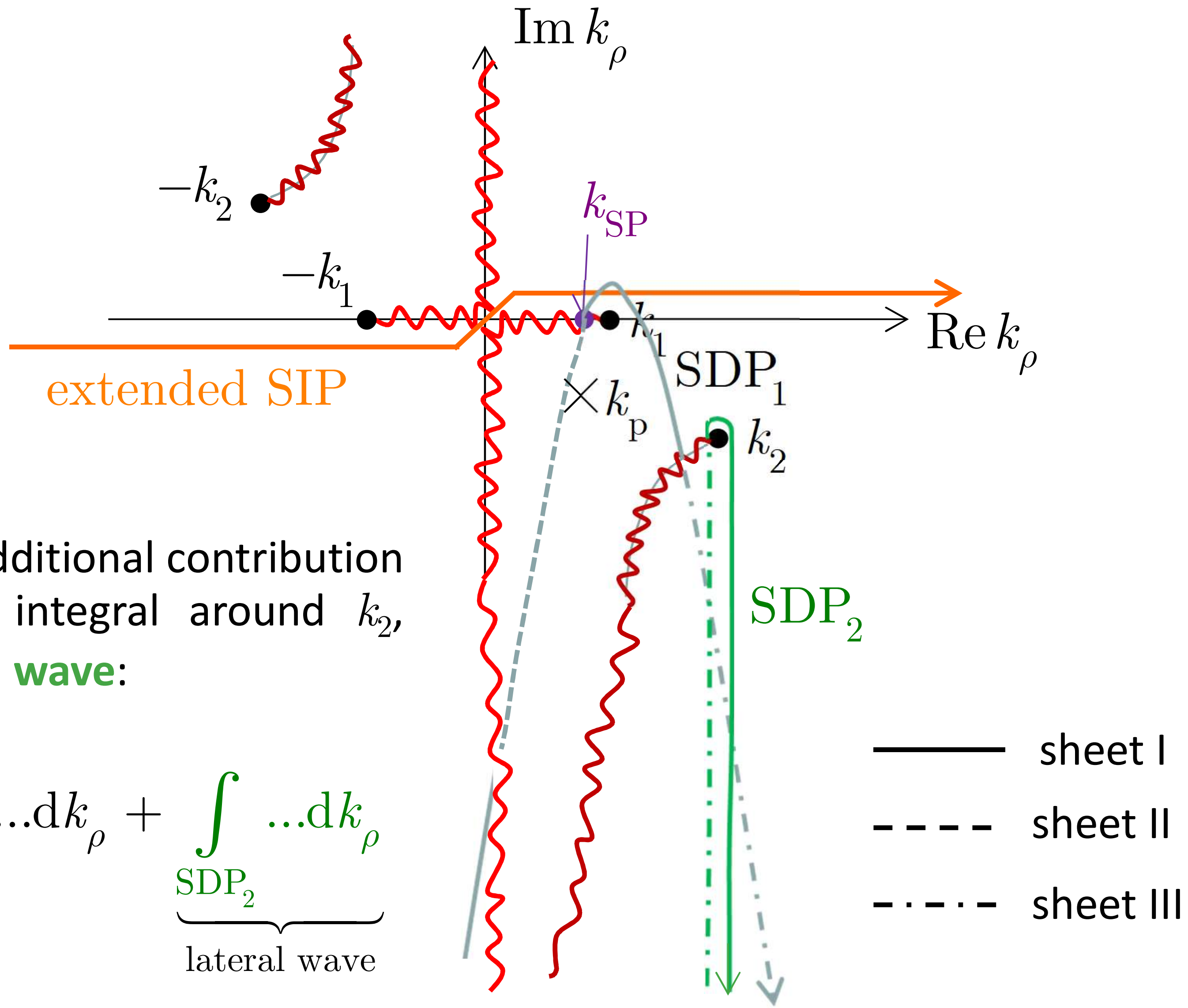


Here $\int_{\infty e^{-j\pi}}^{\infty} \ldots \mathrm{d}k_\rho = \int_{\mathrm{SDP}} \ldots \mathrm{d}k_\rho$

(the arrow indicates the direction of integration along the SDP)

k_2-SDP Contribution: the Lateral Wave

By increasing θ:

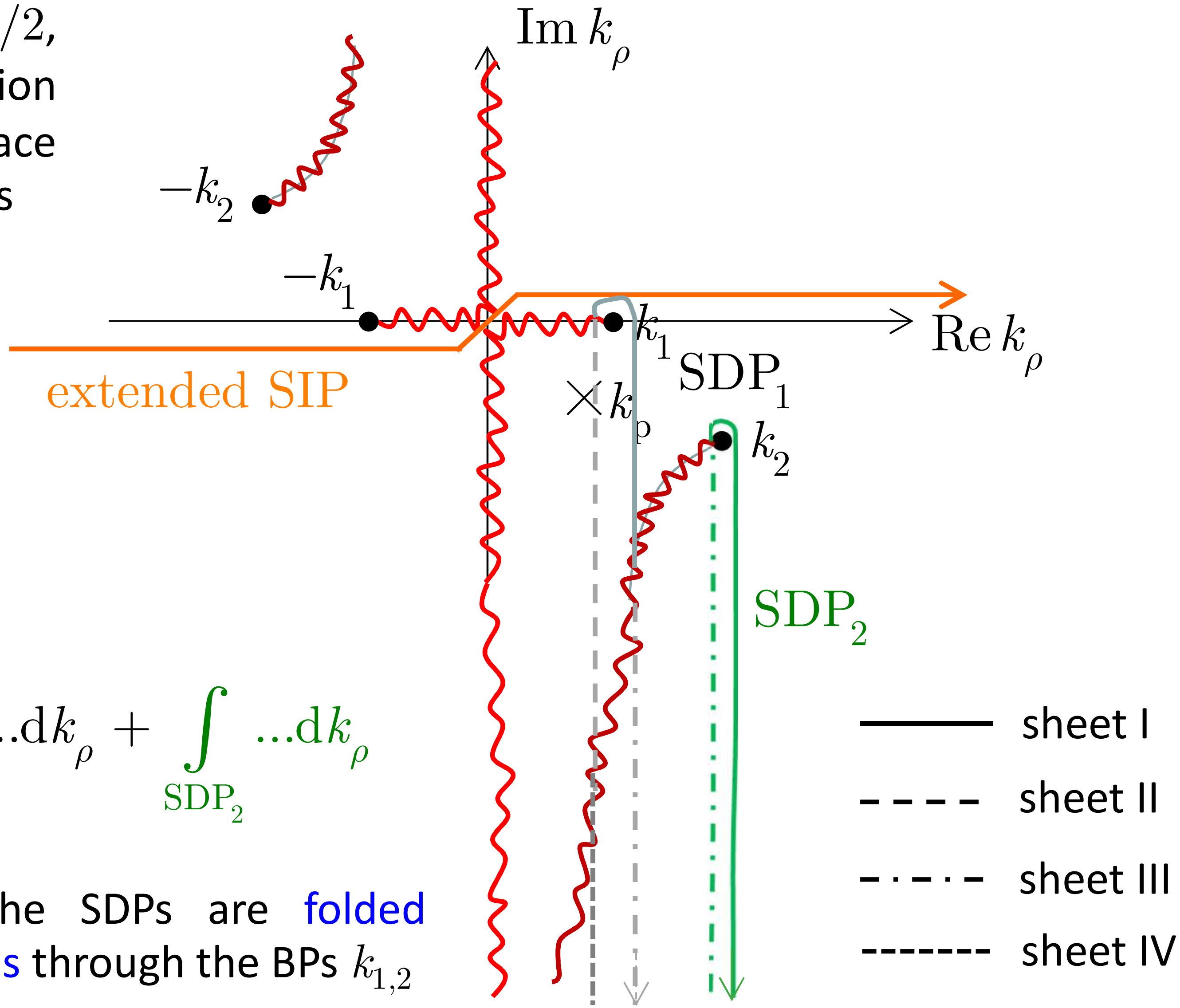


Here we have the additional contribution given by the SDP integral around k_2, known as the **lateral wave**:

$$\int_{\infty e^{-j\pi}}^{\infty} \ldots \mathrm{d}k_\rho = \int_{\mathrm{SDP}_1} \ldots \mathrm{d}k_\rho + \underbrace{\int_{\mathrm{SDP}_2} \ldots \mathrm{d}k_\rho}_{\text{lateral wave}}$$

Steepest-Descent Path: Nonspectral Representation

By letting θ tend to $\pi/2$, i.e., for observation points on the surface $z=0$ at large distances $\rho \gg -z'$:



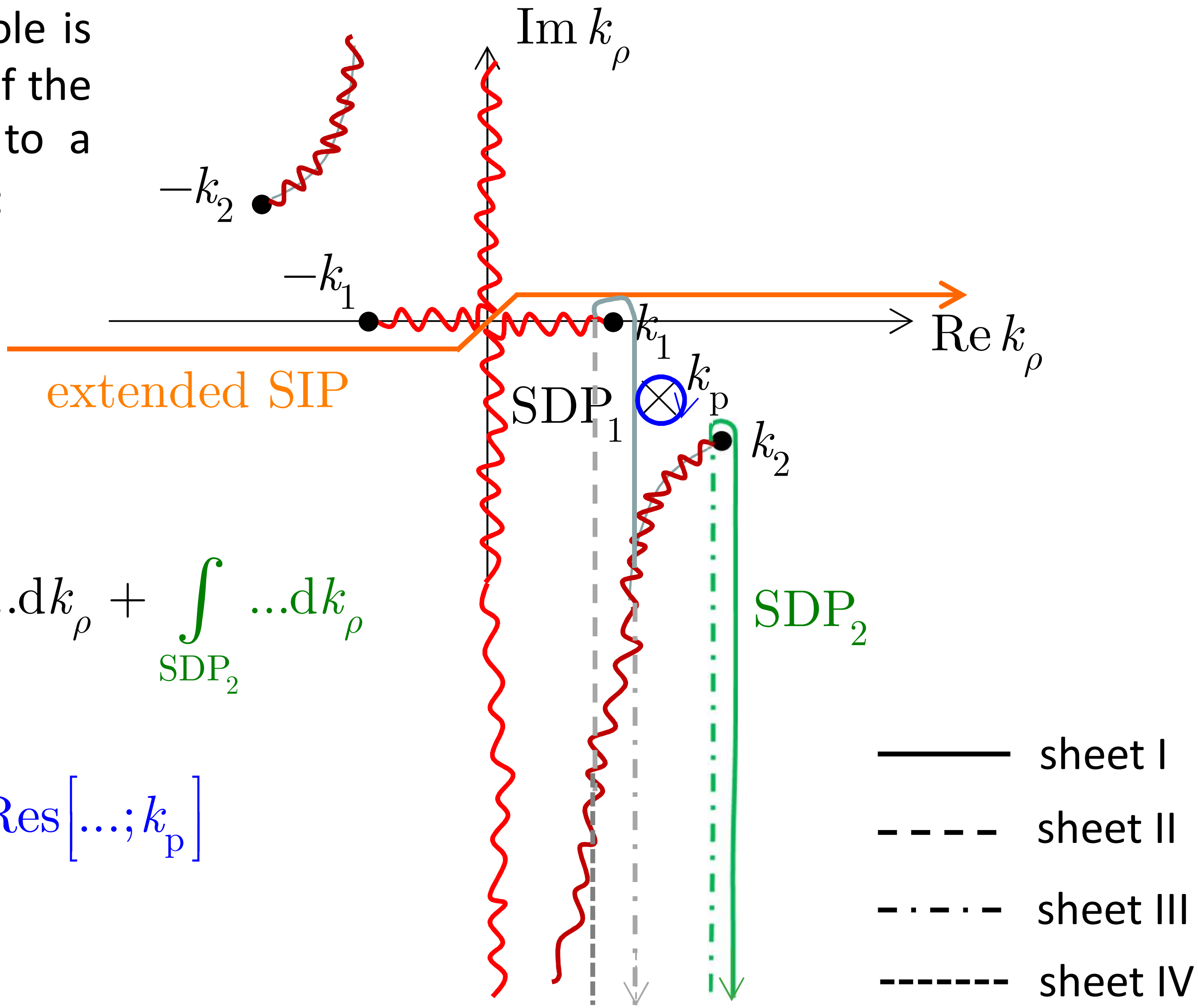
Here

$$\int_{\infty e^{-j\pi}}^{\infty} \ldots \mathrm{d}k_\rho = \int_{\mathrm{SDP}_1} \ldots \mathrm{d}k_\rho + \int_{\mathrm{SDP}_2} \ldots \mathrm{d}k_\rho$$

Note that now the SDPs are folded vertical straight lines through the BPs $k_{1,2}$

Sommerfeld Pole Contribution

If the Sommerfeld pole is located to the right of the BP k_1, it gives rise to a **residue contribution**:



Hence

$$\int_{\infty e^{-j\pi}}^{\infty} \ldots \mathrm{d}k_\rho = \int_{\mathrm{SDP}_1} \ldots \mathrm{d}k_\rho + \int_{\mathrm{SDP}_2} \ldots \mathrm{d}k_\rho$$

$$-2\pi j \,\mathrm{Res}\left[\ldots; k_\mathrm{p}\right]$$

Pole Contribution: Taxonomy

If the dielectric represents a lossy ground at radio frequencies, the Sommerfeld pole is associated to the *Zenneck wave* (**ZW**) and is **never captured** in the nonspectral representation. However, if the dielectric represents, e.g., a metal at THz or optical frequencies, the Sommerfeld pole **may be captured** and is associated to a *Surface Plasmon Polariton* (**SPP**) wave.

	$\varepsilon_r = \varepsilon' - j\varepsilon''$, $\varepsilon'' \geq 0$ (passive medium)	
	$\varepsilon'' \gg \lvert\varepsilon'\rvert$ (high loss)	$\varepsilon'' \ll \lvert\varepsilon'\rvert$ (low loss)
Not captured (*Sommerfeld case*, $\mathrm{Re}k_p < k_1$)	$\varepsilon' > -3/4$ ZW	$\varepsilon' > 3/4$ Brewster mode
Captured (*plasmonic case*, $\mathrm{Re}k_p > k_1$)	$\varepsilon' < -3/4$ SPP	$\varepsilon' < 3/4$ Fano mode

The Role of Zenneck (or SPP) Waves

The nonspectral representation can thus be written as:

$$\int_{\infty e^{-j\pi}}^{\infty} \tilde{f}\left(k_\rho\right) H_0^{(2)}\left(k_\rho \rho\right) k_\rho \mathrm{d}k_\rho = I_1\left\{\tilde{f}\right\} + I_2\left\{\tilde{f}\right\} \underbrace{-2\pi j R_\mathrm{p} H_0^{(2)}\left(k_\mathrm{p}\rho\right) k_\mathrm{p}}_{\substack{\text{absent in the Sommerfeld case} \\ (\mathrm{Re}\, k_\mathrm{p} < k_1)}}$$

$$I_i\left\{\tilde{f}\right\} = \int_{\mathrm{SDP}_i} \tilde{f}\left(k_\rho\right) H_0^{(2)}\left(k_\rho \rho\right) k_\rho \mathrm{d}k_\rho, \qquad i = 1, 2$$

$$R_\mathrm{p} = \mathrm{Res}\left[\tilde{f}\left(k_\rho\right); k_\rho = k_\mathrm{p}\right]$$

In **both** the Sommerfeld and plasmonic cases, if the pole is close to the BP at $k_\rho = k_1$ it gives a **significant contribution** to the value of the BC integral in the nonspectral representation of the surface potential (i.e., at $z=0$), due to its **close topological proximity** to the downward leg of the SDP around k_1.

Nonspectral Representation: SDP Integrals

Upon changing the variable of integration via: $k_\rho = k_i - js^2$

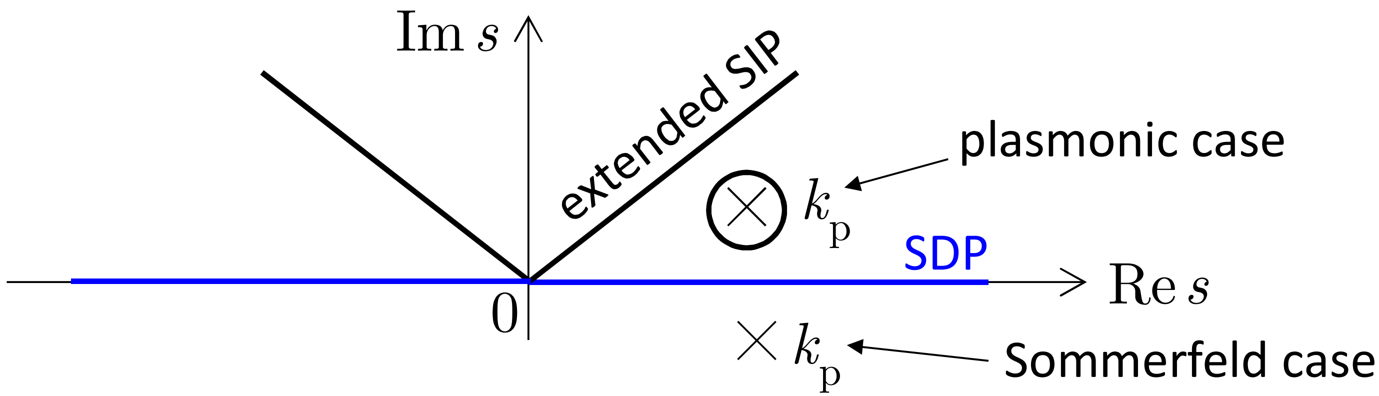
$$I_i\{\tilde{f}\} = 2j\sqrt{\frac{2j}{\pi\rho}}e^{-jk_i\rho}\int_0^{+\infty} F(s)\,e^{-s^2\rho}s\,\mathrm{d}s$$

where

$$F(s) = [\![\tilde{f}(k_\rho)]\!]\sqrt{k_\rho}\mathcal{H}_0^{(2)}(k_\rho\rho) \qquad [\![\tilde{f}(k_\rho)]\!] = \tilde{f}^+(k_\rho) - \tilde{f}^-(k_\rho)$$

Unless the dielectric is very low loss, $\left|e^{-jk_2\rho}\right| \ll \left|e^{-jk_1\rho}\right| = 1$ hence the integral I_2 (i.e., the *lateral wave*) is typically negligible with respect to I_1.

As concerns I_1, the situation in a neighborhood of the origin in the complex s-plane is:



Nonspectral Representation: SDP Integrals

For a reliable numerical evaluation of the SDP integral it is necessary to avoid the spike in the integrand due to the Sommerfeld pole. The way to do this is to **extract the pole**:

$$\int_0^{+\infty} F(s) e^{-s^2\rho} s \, \mathrm{d}s = \underbrace{\int_0^{+\infty} \left[F(s) + \frac{2B_\mathrm{p} s}{s^2 - s_\mathrm{p}^2} \right] e^{-s^2\rho} s \, \mathrm{d}s}_{I_\mathrm{p}} - 2B_\mathrm{p} \underbrace{\int_0^{+\infty} \frac{e^{-s^2\rho}}{s^2 - s_\mathrm{p}^2} s^2 \, \mathrm{d}s}_{I_\mathrm{q}}$$

with

$$s_\mathrm{p} = \sqrt{j\left(k_\mathrm{p} - k_1\right)} \qquad B_\mathrm{p} = \frac{jR_\mathrm{p}}{2s_\mathrm{p}} \sqrt{k_\mathrm{p}} \mathcal{H}_0^{(2)}\left(k_\mathrm{p}\rho\right)$$

In this way:

- the modified integral I_p is **amenable to numerical quadrature** (the integrand is well-behaved, if $\rho \gg -z'$ it is not oscillatory and exponentially decaying (faster for larger ρ);
- the compensating integral I_q has a **closed-form** representation (as shown next)

Nonspectral Representation: SDP$_1$ Integral

As concerns I_{q}, we first transform the integral as

$$I_{\mathrm{q}} = \int_0^{+\infty} e^{-s^2\rho} s\,\mathrm{d}s + s_{\mathrm{p}}^2 \underbrace{\int_0^{+\infty} \frac{e^{-s^2\rho}}{s^2 - s_{\mathrm{p}}^2} s^2\,\mathrm{d}s}_{I_{\mathrm{s}}} = \frac{1}{2}\sqrt{\frac{\pi}{\rho}} + s_{\mathrm{p}}^2 I_{\mathrm{s}}$$

Now I_{s} can be expressed in terms of the **Faddeeva function** $w(z) = e^{-z^2}\operatorname{erfc}(-jz)$ which admits the representation

$$w(z) = \frac{1}{\pi j}\int_{-\infty}^{+\infty} \frac{e^{-t^2}}{t - z}\,\mathrm{d}t = \frac{2z}{\pi j}\int_0^{+\infty} \frac{e^{-t^2}}{t^2 - z^2}\,\mathrm{d}t, \quad \mathrm{Im}\, z > 0$$

upon letting $t = s\sqrt{\rho}$ and $z = s_{\mathrm{p}}\sqrt{\rho}$.

However, since $\mathrm{Im}\, s_{\mathrm{p}}$ may be negative (Sommerfeld case), it is necessary to extend the validity of the above representation by adding to the integral $2\pi j$ times the pole residue when $\mathrm{Im}\, z < 0$ (or half of the pole residue when $\mathrm{Im}\, z = 0$).

Nonspectral Representation: SDP_1 Integral

One eventually finds

$$\boxed{I_{\mathrm{q}} = \frac{1}{2}\sqrt{\frac{\pi}{\rho}}\mathcal{F}\left(p\right)}$$

where

$$p = s_{\mathrm{p}}^2 \rho = j\left(k_{\mathrm{p}} - k_1\right)\rho \quad \text{('numerical distance')}$$

and

$$\mathcal{F}\left(p\right) = 1 + j\sqrt{\pi p}\begin{cases} w\left(\sqrt{p}\right), & \operatorname{Im} s_{\mathrm{p}} > 0 \quad \text{(plasmonic case)} \\ w\left(\sqrt{p}\right) - e^{-p}, & \operatorname{Im} s_{\mathrm{p}} = 0 \quad \text{(transition point)} \\ -w\left(-\sqrt{p}\right), & \operatorname{Im} s_{\mathrm{p}} < 0 \quad \text{(Sommerfeld case)} \end{cases}$$

('attenuation function')

Nonspectral Representation: SDP$_1$ Integral

Collecting these results, we find the final expression for the SDP$_1$ integral:

$$I_1\left\{\tilde{f}\right\} = 2j\sqrt{\frac{2j}{\pi\rho}}e^{-jk_1\rho}\int_0^{+\infty} F\left(s\right)e^{-s^2\rho}s\,\mathrm{d}s$$

$$= \boxed{2j\sqrt{2j}\left[\sqrt{\frac{\rho}{\pi}}I_{\mathrm{p}} - B_{\mathrm{p}}\mathcal{F}\left(p\right)\right]\frac{e^{-jk_1\rho}}{\rho}}$$

It is important to note that $\mathcal{F}(p)$ is **discontinuous** when the sign of $\mathrm{Im}\,s_{\mathrm{p}}$ changes as the Sommerfeld pole crosses the SDP path (i.e., the Sommerfeld case turns into the plasmonic case).

However, the resulting discontinuity in the SDP integral is **exactly compensated** by the residue term that must now be included in the nonspectral representation.

Asymptotic Evaluation of the SDP_1 Integral

The same pole-extraction approach is used for the **asymptotic evaluation** of the SDP integral for $\rho \gg -z'$.

As concerns the I_p integral:

$$I_\mathrm{p} \sim \left[F'\left(0\right) - \frac{2B_\mathrm{p}}{s_\mathrm{p}^2}\right] \int_0^{+\infty} e^{-s^2\rho} s^2 \, \mathrm{d}s = \left[F'\left(0\right) - \frac{2B_\mathrm{p}}{s_\mathrm{p}^2}\right] \frac{1}{4\rho} \sqrt{\frac{\pi}{\rho}}$$

where

$$F'\left(0\right) = \sqrt{k_1} \mathcal{H}_0^{(2)}\left(k_1\rho\right) \lim_{s\to 0} \left. \frac{\left[\!\left[\tilde{f}\left(k_\rho\right) \right]\!\right]}{s} \right|_{k_\rho = k_1 - js^2}$$

As concerns the function $\mathcal{F}(p)$ it can be shown that:

$$\mathcal{F}\left(p\right) \sim -\left(\frac{1}{2p} + \frac{3}{4p^2} + \ldots\right), \quad \left|p\right| \gg 1$$

Asymptotic Evaluation of the SDP_1 Integral

Therefore:

$$I_1\left\{\tilde{f}\right\} = 2j\sqrt{2j}\left[\sqrt{\frac{\rho}{\pi}}I_{\mathrm{p}} - B_{\mathrm{p}}\mathcal{F}\left(p\right)\right]\frac{e^{-jk_1\rho}}{\rho}$$

$$\sim \boxed{-2j\sqrt{2j}\left[-\frac{F'\left(0\right)}{4\rho} + B_{\mathrm{p}}\breve{\mathcal{F}}\left(p\right)\right]\frac{e^{-jk_1\rho}}{\rho}}$$

in terms of the *modified* attenuation function

$$\breve{\mathcal{F}}\left(p\right) = \mathcal{F}\left(p\right) + \frac{1}{2p} = O\left(p^{-2}\right), \;\; \left|p\right| \gg 1$$

so that the resulting asymptotic expansion is accurate to order ρ^{-2} .

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