Tabella riassuntiva dell'aritmetica "estesa" dei limiti.

$$\left.\begin{array}{l} \text{"}\pm\infty + \ell = \pm\infty \quad \forall \ell \in \mathbb{R}\text{"} \\ +\infty + \infty = +\infty \\ -\infty - \infty = -\infty \end{array}\right\} \text{somma.}$$

$$\left.\begin{array}{l} \pm\infty \cdot \ell = \begin{cases} \pm\infty & \text{se } \ell > 0 \\ \mp\infty & \text{se } \ell < 0 \end{cases} \\ (+\infty)\cdot(+\infty) = +\infty \\ (+\infty)\cdot(-\infty) = -\infty \\ (-\infty)\cdot(-\infty) = +\infty \end{array}\right\} \text{prodotto}$$

$$\left.\begin{array}{l} \dfrac{\ell}{\pm\infty} = \begin{cases} 0^{\pm} & \text{se } \ell > 0 \\ 0^{\mp} & \text{se } \ell < 0 \end{cases} \\ \dfrac{\ell}{0^{\pm}} = \begin{cases} \pm\infty & \text{se } \ell > 0 \\ \mp\infty & \text{se } \ell < 0 \end{cases} \\ \dfrac{\pm\infty}{0^{+}} = \pm\infty \\ \dfrac{\pm\infty}{0^{-}} = \mp\infty \end{array}\right\} \text{rapporto.}$$

Forme indeterminate $+\infty - \infty$, $0\cdot(\pm\infty)$, $\dfrac{0}{0}$, $\dfrac{\pm\infty}{\pm\infty}$

$$\lim_{n\to+\infty} \left(n^{\alpha-1} - n^{-\alpha-1}\right) \qquad \alpha \in \mathbb{R}.$$

$$n^{\alpha-1} \to \begin{cases} +\infty & \text{se } \alpha > 1 \\ 1 \ \text{(vale } 1\text{)}! & \text{se } \alpha = 1 \\ 0 & \text{se } \alpha < 1 \end{cases}$$

$$n^{-\alpha-1} \longrightarrow \begin{cases} +\infty & \text{se } \alpha < -1 \\ \equiv 1 & \text{se } \alpha = -1. \\ 0 & \text{se } \alpha > -1 \end{cases}$$

$-\alpha - 1 > 0 \iff \alpha < -1$

$$n^{\alpha-1} - n^{-\alpha-1}$$

se $\alpha > 1$ $\qquad \underbrace{n^{\alpha-1}}_{\to +\infty} - \underbrace{n^{-\alpha-1}}_{\to 0} \longrightarrow +\infty$

$\alpha = 1$ $\qquad 1 - \underbrace{n^{-2}}_{\to 0} \longrightarrow 1$

$-1 < \alpha < 1$ $\qquad \underbrace{n^{\alpha-1}}_{\to 0} - \underbrace{n^{-\alpha-1}}_{\to 0} \longrightarrow 0$

$\alpha = -1$ $\qquad \underbrace{n^{-2}}_{\to 0} - 1 \longrightarrow -1$

$\alpha < -1$ $\qquad \underbrace{n^{\alpha-1}}_{\to 0} - \underbrace{n^{-\alpha-1}}_{\to +\infty} \longrightarrow -\infty$

$$\left(\frac{1}{2} + \frac{1}{3n}\right)^{1-n}$$

E' una successione della forma $\{(a_n)^{b_n}\}$
dove $\{a_n\}, \{b_n\}$ sono generiche successioni di reali,
e $a_n > 0$

Si usa la seguente trasformazione:
si fissa $b > 0$, $b \neq 1$. (per es. $b = 2$, $b = 10$, $b = e$)

$e = 2{,}718\ldots$

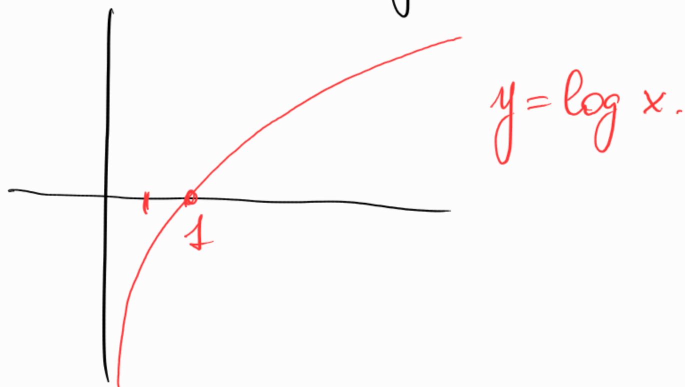
$$\underbrace{(a_n)^{b_n}} = b^{\log_b\left((a_n)^{b_n}\right)} = b^{b_n \log_b(a_n)}$$

(se $x > 0$, $x = b^{\log_b x}$)

Adesso studio l'esponente. $b_n \log_b(a_n)$

Scegliamo $b = e \sim 2{,}718\ldots > 1$ $\log_e x = \ln x = \log x$

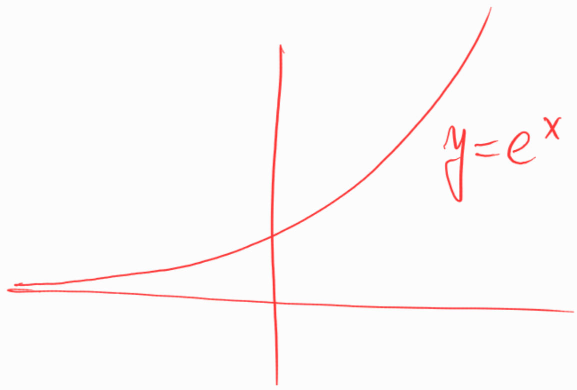
$y = \log x$



$$\left(\frac{1}{2} + \frac{1}{3n}\right)^{1-n} = e^{\underbrace{(1-n)\log\left(\frac{1}{2} + \frac{1}{3n}\right)}_{\to +\infty}} \longrightarrow +\infty.$$

$$\underbrace{(1-n)}_{\to -\infty} \underbrace{\log\left(\frac{1}{2} + \underbrace{\frac{1}{3n}}\right)}_{1/2} \longrightarrow +\infty$$

$\log(1/2) = -\log 2 < 0$



OSS di fatto abbiamo dimostrato che

$(a_n)^{b_n}$ dove $a_n \to 1/2$, $b_n \to -\infty$

$\Rightarrow (a_n)^{b_n} \longrightarrow +\infty$

Abbiamo in realtà provato che

$$\boxed{\begin{array}{ll} \text{"} \ell^{-\infty} = +\infty \text{"} & \forall \ell \in (0,1) \\ \text{"} \ell^{-\infty} = 0 \text{"} & \forall \ell > 1. \end{array}}$$

[cioè $(a_n)^{b_n}$ dove $b_n \to -\infty$, $a_n \to \ell > 1$

$$(a_n)^{b_n} = e^{\boxed{\underbrace{b_n}_{-\infty} \underbrace{\log(a_n)}_{\log \ell > 0}} \to -\infty} \longrightarrow 0$$]

$$\boxed{\begin{array}{l} \text{"} \ell^{+\infty} = \begin{cases} +\infty & \text{se } \ell > 1 \\ 0 & \text{se } 0 < \ell < 1 \end{cases} \text{"} \\ \text{"} 0^{+\infty} = 0 \text{"} \end{array}}$$

$a_n \to 0^+$
$b_n \to +\infty$

$$(a_n)^{b_n} = e^{\overbrace{\underbrace{b_n}_{+\infty} \underbrace{\log a_n}_{-\infty}}^{\to -\infty}} \longrightarrow 0$$

$$\boxed{\begin{array}{l} \text{"} 0^{-\infty} = +\infty \text{"} \\ \text{"} +\infty^{+\infty} = +\infty \text{"} \\ \text{"} +\infty^{-\infty} = 0 \text{"} \end{array}}$$

$$\boxed{\left.\begin{array}{l} a_n \to \ell \in (0,+\infty) \\ b_n \to m \in \mathbb{R} \end{array}\right| \Rightarrow (a_n)^{b_n} \to \ell^{m} = e^{\overset{b_n}{\underset{m}{\downarrow}}\ \overset{\log a_n}{\underset{\log \ell}{\downarrow}}} \to e^{m \log \ell}}$$

Quali sono i casi indeterminati <u>in generale</u>?

$$(a_n)^{b_n} = e^{b_n \log a_n}$$

sono i casi in cui $b_n \log a_n$ è una forma indeterminata

sono cioè quelli in cui viene fuori una forma indeterminata del tipo $0 \cdot \pm\infty$

$$\boxed{\begin{array}{c} (+\infty)^0 \\[1em] 0^0 \\[1em] 1^{\pm\infty} \end{array}}$$

cioè $a_n \to +\infty$, $b_n \to 0$

$a_n \to 0^+$, $b_n \to 0$

$a_n \to 1 \Rightarrow \log a_n \to 0$

$b_n \to \pm\infty$

se $b_n \to 0^-$, allora $\dfrac{1}{b_n} \to -\infty$.

Hyp. $\forall \varepsilon > 0 \quad -\varepsilon < b_n < 0$ def^{te}.

Th. $\forall M > 0$. $\quad \dfrac{1}{b_n} < -M$ def^{te}.

Risoluzione delle forme indeterminate.

$$\lim_{n\to+\infty} (\underbrace{n^2}_{+\infty} - 7 - 3\underbrace{n^4}_{+\infty}) = (+\infty - \infty) = -\infty$$

$$n^2 - 7 - 3n^4 = \underbrace{n^4}_{+\infty}\underbrace{\left(\underbrace{\frac{1}{n^2}}_{0} - \underbrace{\frac{7}{n^4}}_{0} - 3\right)}_{-3} \longrightarrow -\infty$$

Questo si estende a tutti i limiti di polinomi.

$$\lim_{n\to+\infty} P_k(n) \quad , \text{ dove}$$

$$P_k(n) = a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0 =$$

$$= \sum_{j=0}^{k} a_j n^j \quad \text{dove } a_0, \dots a_k \text{ sono reali fissati.}$$

$$a_k \neq 0$$

$$P_k(n) = n^k \underbrace{\left(a_k + \underbrace{\frac{a_{k-1}}{n}}_{0} + \underbrace{\frac{a_{k-2}}{n^2}}_{0} + \dots + \underbrace{\frac{a_0}{n^k}}_{0}\right)}_{a_k}$$

n

$$\boxed{\lim_{n\to+\infty} P_k(n) = \begin{cases} +\infty & \text{se } a_k > 0 \\ -\infty & \text{se } a_k < 0 \end{cases}}$$

$$\lim_{n\to+\infty} \overbrace{\frac{3n^2-2n+5}{-n^2+n+1}}^{\to+\infty} = \left(\frac{+\infty}{-\infty}\right)$$

$$= \lim_{n\to+\infty} \frac{\cancel{n^2}\,\overbrace{\left(3-\frac{2}{n}+\frac{5}{n^2}\right)}^{\to 3}}{\cancel{n^2}\,\underbrace{\left(-1+\frac{1}{n}+\frac{1}{n^2}\right)}_{\to -1}} = -3$$

$-\infty$

$$\lim_{n\to+\infty} \frac{3n^3-2n+5}{-n^2+n+1} = \left(\frac{+\infty}{-\infty}\right) =$$

$$= \lim_{n\to+\infty} \frac{\overset{\nwarrow +\infty}{n^{\cancel{3}}}\,\overbrace{(3+\text{succ}^{ne}\text{ che tende a zero})}^{\to 3}}{\cancel{n^2}\,\underbrace{(-1+\text{succ}^{ne}\text{ che tende a zero})}_{\to -1}} = -\infty$$

$$\lim_{n\to+\infty} \frac{3n^2-2n+5}{-n^3+n+1} = \left(\frac{+\infty}{-\infty}\right) =$$

$$= \lim_{n\to+\infty} \frac{\cancel{n^2}\,(3+\text{succ}^{ne}\text{ che tende a zero})}{n^{\cancel{3}}\,(-1+\text{succ}^{ne}\text{ che tende a zero})} = 0$$

Se serve,
il limite in realtà è 0^-

Se $a_n = \dfrac{P_k(n)}{Q_h(n)}$, dove $P_k(n)$ e $Q_h(n)$ sono polinomi di grado k, h risp.

$$P_k(n) = \sum_{j=0}^{k} a_j n^j$$

$$Q_h(n) = \sum_{j=0}^{h} b_j n_j$$

dove $a_0, \dots a_k, b_0, \dots b_h \in \mathbb{R}$, a_k e $b_h \neq 0$.

$$\lim_{n\to+\infty} \frac{P_k(n)}{Q_h(n)} = \lim_{n\to+\infty} \frac{n^k}{n^h} \boxed{\frac{(a_k + \text{succ}^{ne} \text{ che tende a zero})}{(b_h + \text{succ}^{ne} \text{ che tende a zero})}}$$

$$\downarrow \frac{a_k}{b_h}$$

Risultati:

se $k = h$, il limite vale $\dfrac{a_k}{b_k}$

se $k > h$, il limite viene $\pm\infty$ a seconda del segno di $\dfrac{a_k}{b_h}$

se $k < h$, il limite viene 0
(in realtà è $0^{\pm}$ a seconda del segno di $\dfrac{a_k}{b_h}$)

In sostanza
$$\lim_{n\to+\infty} \frac{P_k(n)}{Q_h(n)} = \lim_{n\to+\infty} \frac{a_k n^k}{b_h n^h}$$

Notazione. Se $\{a_n\}$ è una succ^{ne} infinitesima, cioè $\lim_{n\to+\infty} a_n = 0$, allora scriveremo

$$a_n = o(1) \quad \text{per } n\to+\infty$$

$$\frac{1}{n} = o(1) \quad \text{per } n\to+\infty$$

$$\frac{5n}{n^2+1} = o(1) \quad \text{per } n\to+\infty$$

Attenzione all'uso (in questo caso scorretto) dell'uguale dalle ultime righe, per es., non segue che $\frac{1}{n} = \frac{5n}{n^2+7}$

Sarebbe più corretto $\frac{1}{n} \in o(1)$, ma noi useremo l'uguale.

Ora lo useremo sistematicamente.

$$\lim_{n\to+\infty} \frac{n^2-5n+1}{n-7} = \lim_{n\to+\infty} \frac{\cancel{n^2}}{\cancel{n}} \boxed{\frac{(1+o(1))}{(1+o(1))}} = +\infty$$

$$\downarrow 1$$

<u>OSS.</u> Invece di dire che $a_n \to 6$ per $n\to+\infty$, scriverò

$$a_n = 6 + \underbrace{(a_n-6)}_{= o(1)} = 6 + o(1)$$

$a_n = \ell + o(1)$ per $n\to+\infty$ significa $a_n \to \ell$ $(\ell \in \mathbb{R})$

$$\lim_{n\to+\infty} \frac{n^\alpha - n}{n^\pi - n^3} \qquad \alpha \in \mathbb{R}$$

$$n^\pi - n^3 = (+\infty - \infty) = n^\pi\left(1 - \frac{n^3}{n^\pi}\right) = n^\pi(1+o(1))$$

$$\frac{n^3}{n^\pi} = \frac{1}{n^{\pi-3}} \to 0$$

$$n^\alpha - n = \begin{cases} n^\alpha(1-\underbrace{n^{1-\alpha}}_{\to 0}) = n^\alpha(1+o(1)) & \alpha > 1 \\ = 0 & \alpha = 1 \\ n(\underbrace{n^{\alpha-1}}_{\to 0} - 1) = n(-1+o(1)) & \alpha < 1 \end{cases}$$

$\boxed{\alpha > 1}$

$$\underbrace{\frac{n^\alpha - n}{n^\pi - n^3}}_{a_n} = \frac{n^\alpha \left(1+o(1)\right)}{n^\pi \left(1+o(1)\right)} = n^{\alpha-\pi}\left(1+o(1)\right) \to$$

$$\to \begin{cases} +\infty & \text{se } \alpha > \pi \\ 1 & \text{se } \alpha = \pi \\ 0 & \text{se } 1 < \alpha < \pi \end{cases}$$

$\boxed{\alpha = 1}$ $\quad a_n \equiv 0 \to 0$

$\boxed{\alpha < 1}$ $\quad a_n = \dfrac{\cancel{n}\left(-1+o(1)\right)}{n^{\pi-1}\left(1+o(1)\right)} \longrightarrow 0.$

Riferimento sul testo consigliato: §§ 2.6, 2.7, 2.8.