Domani: Lezione $10 \to 13$ Aula Bianchi Bandinelli.
Laboratorio Mat. $14 \to 15$ Aula 14

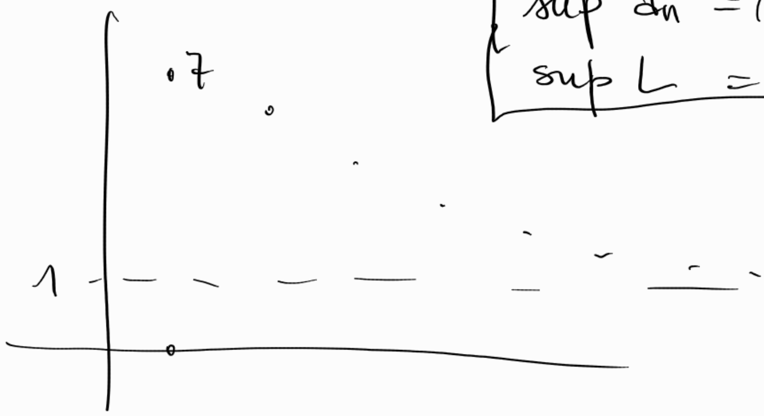
$$L = \left\{ \underbrace{\frac{n^2+5n+1}{n^2}}_{1+\frac{5}{n}+\frac{1}{n^2} = a_n} ; \quad n = 1, 2, \ldots \right\}.$$ Trovare sup e inf.

OSS a_n decrescente strett.

$\Rightarrow \quad a_{n+1} < a_n \quad \forall n \in \mathbb{N}$.

Congettura $\inf L = 1$,

Poiché a_n è decrescente,
$\sup a_n = \max a_n = a_1 = 7$
$\sup L = \max L = 7$



$7 = \max L$.

1) 7 è un maggiorante di L? $\quad 7 \geq a_n \quad \forall n = 1, 2 \ldots$

2) $\forall \varepsilon > 0 \ \exists x \in L$ t.c. $x > 7 - \varepsilon$. Ovvio, basta prendere $x = 7 \in L$.

Proviamo che $\inf L = 1$.

1) $\forall x \in L \quad x \overset{?}{\geq} 1 \iff \forall n \in \mathbb{N}_+ \quad \cancel{1} + \frac{5}{n} + \frac{1}{n^2} \overset{?}{\geq} \cancel{1}$ OK.

2) $\forall \varepsilon > 0 \quad \exists x \in L$ t.c. $1 + \varepsilon > x$

$\exists n \in \mathbb{N}_+$ t.c. $\cancel{1} + \varepsilon > \cancel{1} + \frac{5}{n} + \frac{1}{n^2}$

$(*)$

1° modo $\boxed{\varepsilon n^2 > 5n+1}$

$\varepsilon n^2 - 5n - 1 > 0.$

$\varepsilon x^2 - 5x - 1 = 0 \qquad x_{1,2} = \frac{5 \pm \sqrt{25+4\varepsilon}}{2\varepsilon}$

In particolare (*) è vera per $n > \frac{5+\sqrt{25+4\varepsilon}}{2\varepsilon}$

Sicuramente esiste n t.c. $n > \frac{5+\sqrt{25+4\varepsilon}}{2\varepsilon}$

2° modo devo trovare n t.c. $\frac{5}{n} + \frac{1}{n^2} < \varepsilon.$

si ha $\frac{5}{n} + \frac{1}{n^2} \leq \frac{5}{n} + \frac{1}{n} = \frac{6}{n} < \varepsilon$

$\left[\frac{1}{n^2} \leq \frac{1}{n}\right]$ $\qquad \boxed{n > \frac{6}{\varepsilon}}$

$K = \left\{ \frac{n}{n^2+20}, \; n \in \mathbb{N}_+ \right\}$

a_n

$a_n > 0$

$a_1 = \frac{1}{21}, \quad a_2 = \frac{2}{24} = \frac{1}{12}, \quad a_3 = \frac{3}{29},$

$a_4 = \frac{4}{36} = \frac{1}{9}, \quad a_5 = \frac{5}{45} = \frac{1}{9}, \quad a_6 = \frac{6}{56} = \frac{3}{28}$

congetture: $\inf K = 0$

$\sup K = \max K = a_4 = a_5 = \frac{1}{9}.$

Studiamo la crescenza/decrescenza di $\{a_n\}$

$a_{n+1} \underset{?}{\overset{?}{\leq}} a_n$?

$$\frac{n+1}{\underbrace{(n+1)^2+20}_{n^2+2n+21}} \overset{?}{<} \frac{n}{n^2+20}$$ moltiplico per $(n^2+20)(n^2+2n+21)$ >0

$$(n+1)(n^2+20) \overset{?}{<} n\,(n^2+2n+21)$$

$$\cancel{n^3}+20n+n^2+20 \overset{?}{<} \cancel{n^3}+2n^2+21n$$

$$n^2+n-20 \overset{?}{>} 0 \qquad \text{vero } \Big|\, \forall n>4$$

$$x^2+x-20=0 \iff x=-5 \ \vee\ x=4$$

Per $n \geq 5$ $\qquad a_{n+1} < a_n$

$n=4$ $\qquad a_{n+1} = a_n$

$n<4$ $\qquad a_{n+1} > a_n$

La successione cresce fino ad $a_4 = a_5 = 1/9$, poi decresce

$a_4 = a_5 = 1/9$ è il massimo della successione.

Verifichiamo che $\inf a_n = 0$.

1) $0 \leq a_n = \dfrac{n}{n^2+20}$ $\forall n$ VERO!

2) Fissato $\varepsilon>0$, devo trovare n t.c. $\boxed{\dfrac{n}{n^2+20} \overset{?}{<} \varepsilon.}$

1° modo $\boxed{n \overset{?}{<} \varepsilon n^2+20\varepsilon.}$

$$\boxed{\varepsilon n^2-n+20\varepsilon>0 \quad (*)}$$

$$\varepsilon x^2-x+20\varepsilon=0 \qquad \boxed{x_{1,2}=\frac{1\pm\sqrt{1-80\varepsilon^2}}{2\varepsilon}.}$$

se $\varepsilon > \dfrac{1}{4\sqrt{5}}$ $\Delta<0$, allora $(*)$ è vera $\forall n$

$\Delta = 0$ se $\varepsilon = \frac{1}{4\sqrt{5}}$, allora $(*)$ è vera $\forall n$

$\Delta > 0$ se $0 < \varepsilon < \frac{1}{4\sqrt{5}}$, allora $(*)$ è vera $\forall n > \frac{1+\sqrt{1-80\varepsilon^2}}{2\varepsilon}$

2° modo   Devo trovare n t.c. $\frac{n}{n^2+20} < \varepsilon$.

$$\frac{n}{n^2+20} \leq \frac{n}{n^2} = \frac{1}{n} < \varepsilon \qquad \text{vera se } \boxed{n > \frac{1}{\varepsilon}}$$

$$\lim_{n\to+\infty} \sqrt{\frac{9n}{n+2}} = 3 \qquad \text{Verifica.}$$

Fix $\varepsilon>0$, cerco k t.c. $\underbrace{\left|\sqrt{\frac{9n}{n+2}}-3\right| < \varepsilon}_{(1)}$ $\forall n>k$

$$\sqrt{\frac{9n}{n+2}} \overset{?}{>} 3 \qquad \cancel{3}\sqrt{\frac{n}{n+2}} \overset{?}{>} \cancel{3}\ 1$$

NO, vale la disug opposta

$$\sqrt{\frac{n}{n+2}} < 1$$

La dis (1) diventa $3-\sqrt{\frac{9n}{n+2}}<\varepsilon$

$$\sqrt{\frac{9n}{n+2}} > \underbrace{3-\varepsilon}_{>0}$$

Posso sempre supporre $0<\varepsilon<3$

$$\frac{9n}{n+2} > (3-\varepsilon)^2 = 9-6\varepsilon+\varepsilon^2$$

$$9n > (n+2)(9-6\varepsilon+\varepsilon^2)$$

$$\cancel{9n} > \cancel{9n} + n\underbrace{(-6\varepsilon+\varepsilon^2)}_{\varepsilon(\varepsilon-6)} + 2(9-6\varepsilon+\varepsilon^2)$$

$$n\,\underbrace{\varepsilon(6-\varepsilon)}_{>0} > 2(9-6\varepsilon+\varepsilon^2)$$

$$n > \frac{2(9-6\varepsilon+\varepsilon^2)}{\varepsilon(6-\varepsilon)} = K.$$

$$\lim_{n\to+\infty} \frac{n^3}{n+5} = +\infty$$

Fix $M>0$, cerco k t.c. $\frac{n^3}{n+5} > M \quad \forall n > k$.

se moltiplico per $n+5$, viene $n^3 > (n+5)M$. diseq^{ne} di 3° grado.

Modo suggerito da uno studente presente:

$$\frac{n^3}{n+5} = \frac{n^3(n-5)}{n^2-25} \geq \frac{n^{\cancel{3}}(n-5)}{\cancel{n^2}} \geq n > M$$

$\boxed{n>5}$ $\quad$ $n \geq 1$ $\quad$ $n>5$, $n \geq 6$ $\quad$ $K = \max\{6, M\}$

Altro modo:

$$\frac{n^3}{n+5} \geq \frac{n^3}{n+n} = \frac{n^{\cancel{3}}}{2\cancel{n}} = \frac{n^2}{2} > M$$

$n+5 \leq n+n$

$5 \leq n$

$n^2 > 2M$

$n > \sqrt{2M}$

$$n > \max\{5, \sqrt{2M}\}$$

Altro modo proposto da uno studente:

$$\frac{n^3}{n+5}$$

$$\begin{array}{l|l} n^3 & n+5 \\ \hline -n^3 - 5n^2 & n^2 - 5n + 25 \\ \hline / \quad -5n^2 & \\ \quad 5n^2 + 25n & \\ \hline / \quad 25n & \\ \quad -25n - 125 & \end{array}$$

$$n^3 = (n+5)(n^2-5n+25) - 125$$

$$\frac{n^3}{n+5} = n^2 - 5n + 25 - \frac{125}{n+5} > M.$$

≥ -25

$$n^2 - 5n + 25 - \frac{125}{n+5} \geq n^2 - 5n + \cancel{25} - \cancel{25} > M.$$

$$n > \frac{5 + \sqrt{25 + 4M}}{2}$$