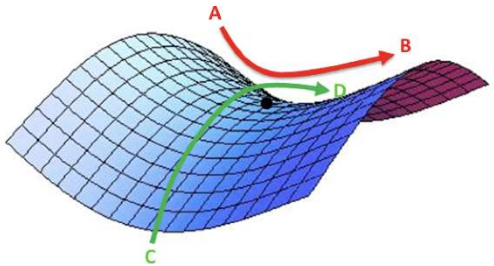
Ph.D. in Information and Communication Engineering

Ph.D. Course on

Analytical Techniques for Wave Phenomena



Lesson 2

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Fundamentals on Complex Functions: Integration

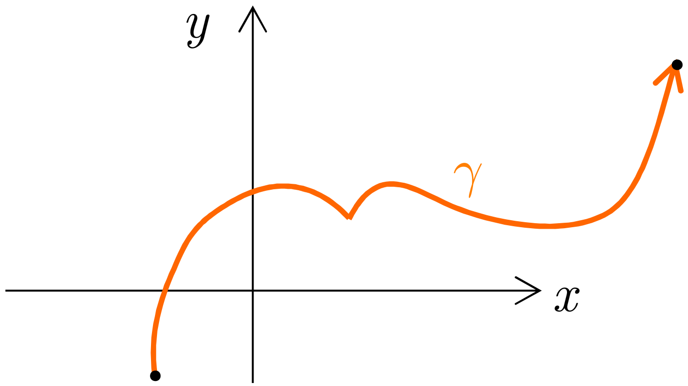
Complex Integration

Many important properties of analytic functions are very difficult to prove without use of complex integration.

As in the real case, we distinguish between:

•**Indefinite integrals**: a function whose derivative equals a given analytic function in a region

•**Definite integrals**: these are taken over *piecewise differentiable* arcs and are not limited to analytic functions but can be defined for continuous functions:



Definite Integral along a Piecewise Differentiable Arc

Definition:

$$\int_{\gamma} f(z)\,\mathrm{d}z \doteq \int_{a}^{b} f\big(z(t)\big)\,z'(t)\,\mathrm{d}t$$

where $z(t)$ is a parametric representation of the arc γ: $\gamma:\ z = z(t),\quad a \le t \le b$

It is readily proved that this definition does not depend on the parameterization of the arc γ. In fact, changing representation through

$$t = \phi(\tau),\quad \alpha \le \tau \le \beta$$
$$a = \phi(\alpha), b = \phi(\beta)$$

$\Rightarrow$

$$\gamma:\ z = \tilde{z}(\tau) = z\big(\phi(\tau)\big),\quad \alpha \le \tau \le \beta$$

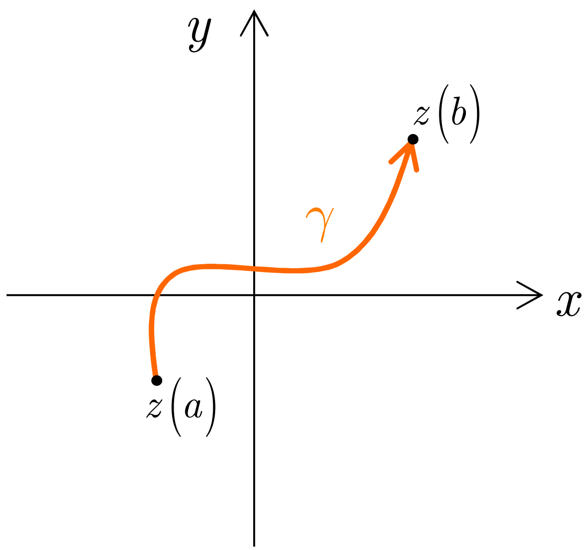
one finds

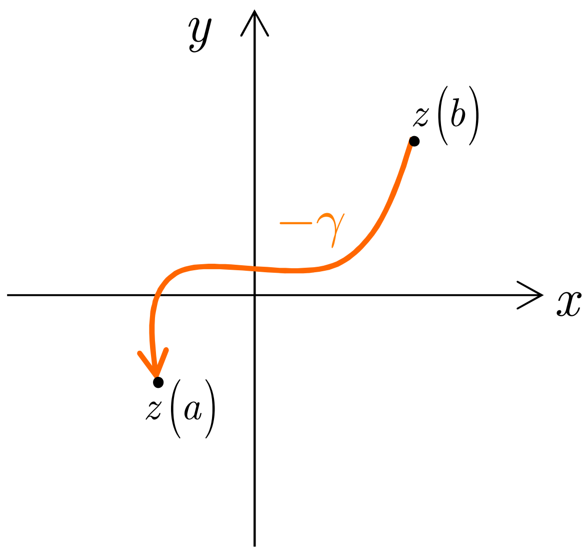
$$\int_{a}^{b} f\big(z(t)\big)\,z'(t)\,\mathrm{d}t = \int_{\alpha}^{\beta} f\Big(z\big(\phi(\tau)\big)\Big)\underbrace{z'\big(\phi(\tau)\big)\phi'(\tau)}_{=\tilde{z}'(\tau)}\,\mathrm{d}\tau = \int_{\alpha}^{\beta} f\big(\tilde{z}(\tau)\big)\,\tilde{z}'(\tau)\,\mathrm{d}\tau$$

Integration over Paths: Elementary Properties

- **Integration along the opposite arc:**

$$\int_{-\gamma} f(z)\,\mathrm{d}z = \int_b^a f(z(t))\,z'(t)\,\mathrm{d}t = -\int_{\gamma} f(z)\,\mathrm{d}z$$

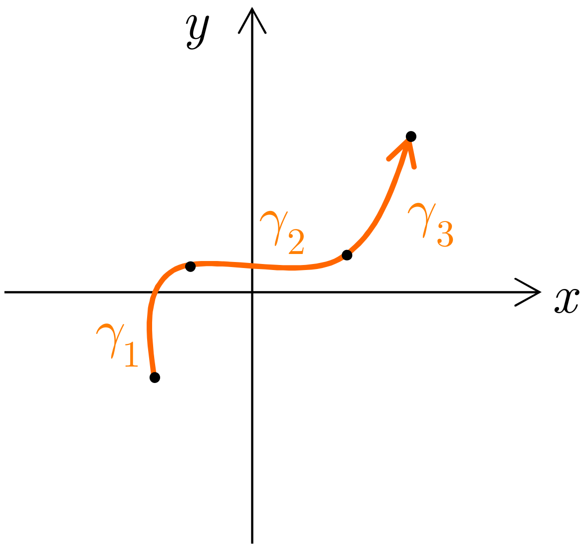




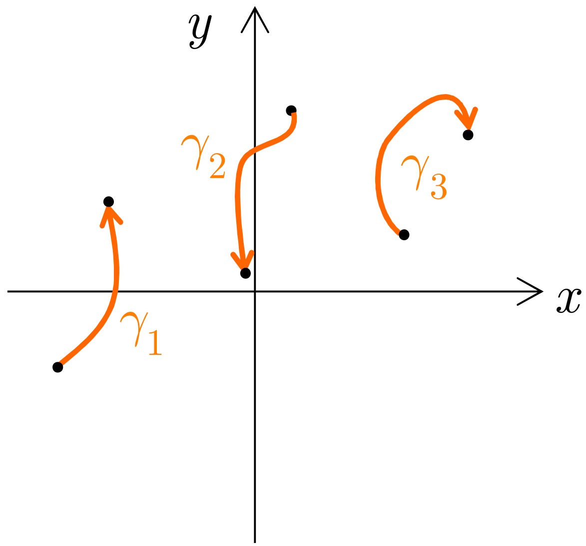
Integration over Paths: Elementary Properties

- **Additivity**:

$$\int_{\gamma_1+\gamma_2+\ldots\gamma_n} f\left(z\right)\mathrm{d}z = \int_{\gamma_1} f\left(z\right)\mathrm{d}z + \ldots + \int_{\gamma_n} f\left(z\right)\mathrm{d}z$$



If the curves γ_i are portions of the same differentiable arc, this is a property of the integral



Otherwise, the RHS defines the meaning of the LHS…

Integration over Paths: Elementary Properties

- **Relation with the** integral w.r.t the arc length

$$\left|\int_\gamma f\,\mathrm{d}z\right| \le \int_\gamma |f||\mathrm{d}z| = \int_a^b \left|f\left(z\left(t\right)\right)\right|\left|z'\left(t\right)\right|\mathrm{d}t = \int_\gamma |f|\,\mathrm{d}s$$

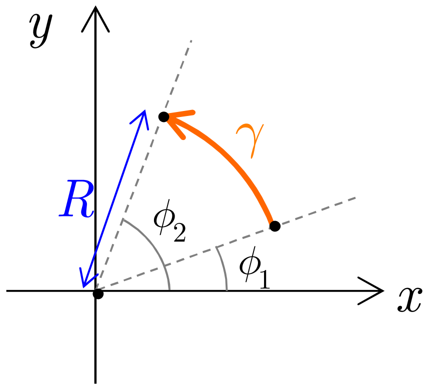
integral w.r.t. the arc length.

Example:

$$f\left(z\right) = e^{jz}$$

$$\gamma:\ z = Re^{j\phi},\ 0 < \phi_1 \le \phi \le \phi_2 < \frac{\pi}{2}$$

$$e^{jz} = e^{jRe^{j\phi}} = e^{-R\sin\phi}e^{jR\cos\phi}$$



$$\left|\int_\gamma e^{jz}\,\mathrm{d}z\right| \le \int_\gamma e^{-R\sin\phi}\left|\mathrm{d}z\right| \le e^{-R\sin\phi_1}\int_\gamma \mathrm{d}s = e^{-R\sin\phi_1}R\left(\phi_2-\phi_1\right) \underset{R\to+\infty}{\longrightarrow} 0$$

Integration over Paths: Elementary Properties

- **Definition in terms of integrals of real differential forms**

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma (u + jv)\,\mathrm{d}(x + jy) = \int_\gamma u\mathrm{d}x - v\mathrm{d}y + j\int_\gamma v\mathrm{d}x + u\mathrm{d}y$$

$$\int_\gamma u\mathrm{d}x - v\mathrm{d}y = \int_a^b \Big\{u\big[x(t), y(t)\big]x'(t) - v\big[x(t), y(t)\big]y'(t)\Big\}\mathrm{d}t$$

$$\int_\gamma v\mathrm{d}x + u\mathrm{d}y = \int_a^b \Big\{v\big[x(t), y(t)\big]x'(t) + u\big[x(t), y(t)\big]y'(t)\Big\}\mathrm{d}t$$

Independence of Path for Given Endpoints

Let us recall the following fundamental theorem for line integrals of (real or complex) differential forms:

Theorem:

The line integral $\int_\gamma p\mathrm{d}x + q\mathrm{d}y$, defined in Ω, depends only on the end points of γ if and only if there exists a function $U(x,y)$ in Ω with $\dfrac{\partial U}{\partial x} = p, \quad \dfrac{\partial U}{\partial y} = q$.

This is of course the well-known fact that a 2D vector field $\mathbf{v}$ with components (p,q) is conservative if and only if it admits a (scalar) potential U (i.e., $\mathbf{v}=\nabla U$).

Independence of Path for Given Endpoints

Applying this to $\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}x + jf(z)\,\mathrm{d}y$ we see that the integral depends only on the endpoints if and only if there exists F such that

$$\frac{\partial F}{\partial x} = f, \quad \frac{\partial F}{\partial y} = jf$$

i.e., iff

$$\frac{\partial F}{\partial x} = -j\frac{\partial F}{\partial y}\text{(Cauchy-Riemann)}$$

(in fact, it it can readily be verified that this is a compact way of writing the Cauchy-Riemann equations that characterize analytic functions),

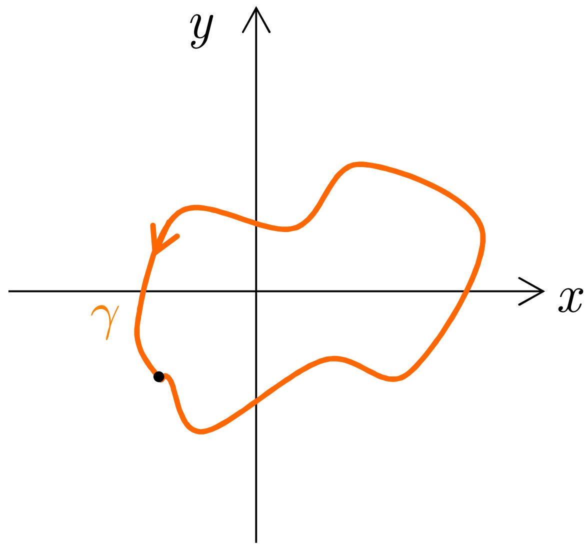
hence if and only if f is the derivative of an analytic function F.

Integrals over Closed Curves

Saying that an integral depends only on the end points is equivalent to saying that **the integral over any closed curve is zero**.

Hence if a continuous f is the derivative of a function F analytic in Ω, then for any **closed** curve γ in Ω

$$\int_{\gamma} f(z)\,\mathrm{d}z = 0$$



and viceversa (under these conditions we shall see that f is itself analytic in Ω).

Integrals over Closed Curves (cont'd)

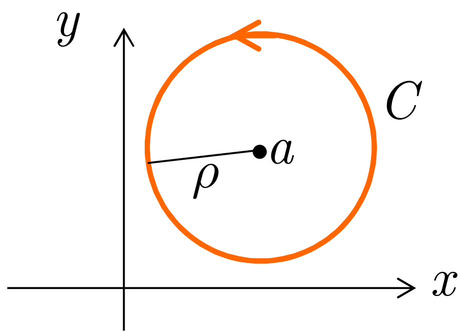
Examples:

- $\int_{\gamma} (z-a)^n \, \mathrm{d}z = 0, \quad n \neq -1$

  In fact, $(z-a)^n$ is the derivative of $\dfrac{1}{n+1}(z-a)^{n+1}$.

- For $n = -1$, the integral over a closed curve **is not always zero**. In fact, let C be the circle $z = a + \rho e^{jt}, 0 \le t \le 2\pi$:

$$\int_{C} \frac{\mathrm{d}z}{z-a} = \int_{0}^{2\pi} j \, \mathrm{d}t = 2\pi j$$



(hence it is impossible to define a single-valued branch of $\log(z-a)$ inside an annulus $\rho_1 \le |z-a| \le \rho_2$)

The Index of a Point w.r.t. a Closed Path

The latter result admits the following important generalization:

Theorem:

Let γ be a closed path, let Ω be the complement of γ and define

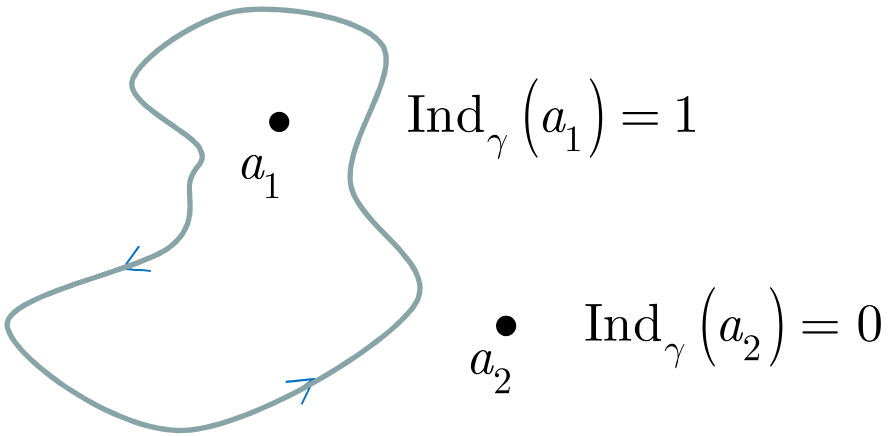
$$\operatorname{Ind}_{\gamma}\left(z\right) = \frac{1}{2\pi j}\int_{\gamma} \frac{\mathrm{d}\zeta}{\zeta - z}$$

Then $\operatorname{Ind}_{\gamma}\left(z\right)$ is an integer-valued function on Ω which is constant in each component of Ω and which is 0 in the unbounded component of Ω.

$\operatorname{Ind}_{\gamma}\left(z\right)$ is called the **index** of z with respect to γ.

The Index of a Point w.r.t. a Closed Path

Examples:



It can be shown that $2\pi\mathrm{Ind}_\gamma(a)$ is the net increase of the argument of $z(t) - a$ as $z(t)$ describes the closed curve γ.

If we divide this increase by 2π we obtain the number of times γ winds around a.

Hence the index is also often termed the **winding number** of γ with respect to a.

The Local Cauchy Theorem and its Consequences

The Local Cauchy Theorem

This is *fundamental*:

Theorem (Cauchy-Goursat)

Suppose Ω is a convex open set, $p \in \Omega$, f is continuous in Ω, $f \in H\left(\Omega \setminus p\right)$.
Then $f = F'$ for some $F \in H\left(\Omega\right)$, hence

$$\int_{\gamma} f\left(z\right) \mathrm{d}z = 0$$

for every closed path γ in Ω.

We shall see that our hypothesis actually implies $f \in H\left(\Omega\right)$, so that the exceptional point p is not really exceptional.

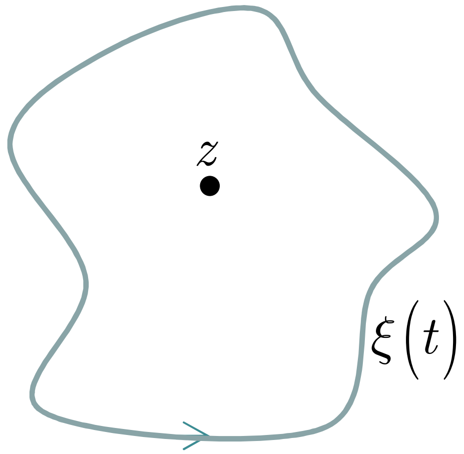
However, the above formulation of the theorem is useful in the proof of the Cauchy Integral Formula…

The Local Cauchy Integral Formula

Theorem (Cauchy formula in a convex set)

Suppose γ is a closed path in a convex open set Ω, and $f \in H(\Omega)$.
If $z \in \Omega$ and $z \notin \gamma$, then

$$f(z) \cdot \mathrm{Ind}_{\gamma}(z) = \frac{1}{2\pi j} \int_{\gamma} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi$$



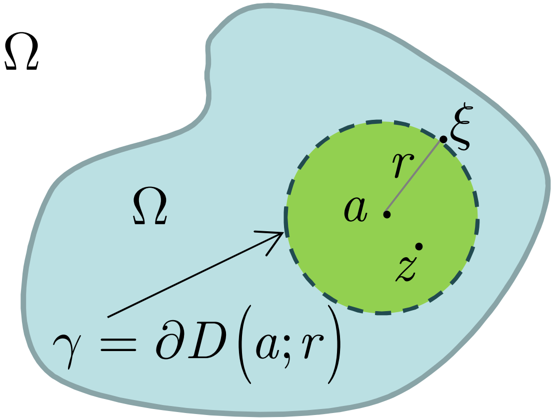
Representability by Power Series

The Cauchy Integral Formula allows for proving the following fundamental

Theorem (representability by power series)

> For every open set Ω, every $f \in H(\Omega)$ is representable by power series in Ω.

In fact, from Cauchy integral formula applied to $D(a;r) \subset \Omega$



$$f(z) = \frac{1}{2\pi j}\int_\gamma \frac{f(\xi)}{\xi - z}\mathrm{d}\xi = \frac{1}{2\pi j}\int_\gamma f(\xi)\sum_{n=0}^{+\infty}\frac{(z-a)^n}{(\xi-a)^{n+1}}\mathrm{d}\xi$$

this geometric series converges uniformly on γ because $\left|\frac{z-a}{\xi-a}\right| < 1$

hence the summation and integration can be interchanged:

$$f(z) = \sum_{n=0}^{+\infty}\underbrace{\frac{1}{2\pi j}\int_\gamma \frac{f(\xi)}{(\xi-a)^{n+1}}\mathrm{d}\xi}_{c_n}(z-a)^n = \sum_{n=0}^{+\infty} c_n (z-a)^n$$

Consequences: Analiticity of the Derivatives

This has an immediate consequence:

For every open set Ω, if $f \in H\left(\Omega\right)$ then $f' \in H\left(\Omega\right)$.

and thus every complex differentiable (i.e., analytic) function is **infinitely differentiable**, each derivative being itself **analytic**.

Contrast this with the behavior of real functions of a real variable…

Consequences: Morera Theorem

The Cauchy theorem has a useful converse, which is a direct consequence of the latter statement:

Theorem (Morera)

Suppose f is a continuous complex function in an open set Ω such that

$$\int_{\gamma} f\left(z\right)\mathrm{d}z = 0$$

for all closed curves γ. Then $f \in H\left(\Omega\right)$.

Proof: The hypothesis implies that $f = F',\ F \in H\left(\Omega\right)$. We now know that f is itself analytic.

Consequences: Zeros of an Analytic Function

Theorem

Let Ω be a nonempty *connected* open set, $f \in H(\Omega)$, and

$$Z(f) = \{a \in \Omega : f(a) = 0\}$$ 'zero set' of f

Then either $Z(f) = \Omega$ or $Z(f)$ has no limit point in Ω. In the latter case there corresponds to each $a \in Z(f)$ a unique positive integer $m = m(a)$ such that

$$f(z) = (z-a)^m g(z), \qquad (z \in \Omega)$$

where $g \in H(\Omega)$ and $g(a) \neq 0$; furthermore, $Z(f)$ is at most countable.

The integer m is called **the order of the zero** which f has at the point a.

Consequently, if $f, g \in H(\Omega)$ and if $f(z) = g(z)$ for all z in some set which has a limit point in Ω, then $f(z) = g(z)$ for all $z \in \Omega$ (a **uniqueness** theorem).

Consequences: Removable Singularities

If $a \in \Omega$ and $f \in H\left(\Omega \setminus a\right)$ then f is said to have an **isolated singularity** at a.

If f can be so defined at a that the extended function is analytic in Ω, then the singularity is said to be **removable**.

This occurs iff f is bounded in $D'\left(a;r\right) \doteq \left\{z : 0 < \left|z-a\right| < r\right\}$ for some r.

Proof:

Define $h\left(a\right) = 0$ and $h\left(z\right) = \left(z-a\right)^2 f\left(z\right)$ in $\Omega \setminus a$. The boundedness assumption implies $h'\left(a\right) = 0$. Since h is evidently differentiable at any other point of Ω, we have $h \in H\left(\Omega\right)$ so

$$h\left(z\right) = \sum_{n=2}^{+\infty} c_n \left(z-a\right)^n, \qquad \left(z \in D\left(a;r\right) = \left\{z : \left|z-a\right| < r\right\}\right)$$

We obtain the desired analytic extension of f by setting $f\left(a\right) = c_2$, for then

$$f\left(z\right) = \sum_{n=0}^{+\infty} c_{n+2} \left(z-a\right)^n, \qquad \left(z \in D\left(a;r\right)\right)$$

Consequences: Classification of Isolated Singularities

Theorem

If $a \in \Omega$ and $f \in H(\Omega \setminus a)$, then one of the following cases must occur:

a) f has a removable singularity at a.

b) There are numbers $c_1, c_2, \ldots, c_m$, where m is a positive integer and $c_m \neq 0$, such that

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$

has a removable singularity at a.

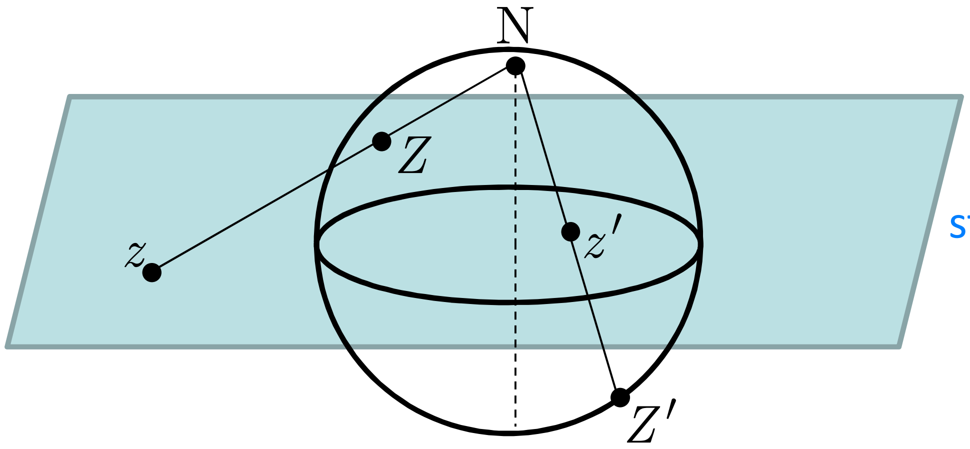
c) If $r > 0$ and $D(a;r) \subset \Omega$ then $f(D'(a;r))$ is dense in the complex plane.

In case b) f is said to have a **pole** of order m at a and $\sum_{k=1}^{m} c_k (z-a)^{-k}$ is called the **principal part** of f at a.

In case c) f is said to have an **essential singularity** at a .

The Point at Infinity of the Complex Plane

For many purposes it is useful to extend the system of complex numbers by introduction of the symbol ∞ to represent infinity. The points in the plane together with the point at infinity form the extended complex plane.



stereographic projection

The notion of isolated singularity applies also to functions analytic in a neighborhood $|z| > R$ of ∞. Since $f(\infty)$ is not defined, we treat ∞ as an isolated singularity and, by convention, it has the same character of removable singularity, pole, or essential singularity as the singularity of $g(z) = f(1/z)$ at $z = 0$.

Representability in Power Series: Cauchy Estimates

We now exploit the fact that the restriction of a power series $\sum c_n (z-a)^n$ to a circle with center at a is a trigonometric series:

$$f(z) = \sum_{n=0}^{+\infty} c_n (z-a)^n \rightarrow f\left(a + re^{j\theta}\right) = \sum_{n=0}^{+\infty} c_n r^n e^{jn\theta}$$

hence

$$c_n r^n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f\left(a + re^{j\theta}\right) e^{-jn\theta} \mathrm{d}\theta \qquad \text{Fourier coefficients}$$

$$\sum_{n=0}^{+\infty} \left|c_n\right|^2 r^{2n} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \left|f\left(a + re^{j\theta}\right)\right|^2 \mathrm{d}\theta \qquad \text{Parseval formula}$$

Consequently, since $c_n = f^{(n)}(a) / n!$, if $f \in H\big(D(a;R)\big)$ and $\left|f(z)\right| \le M$ for all $z \in D(a;R)$, then

$$\left|f^{(n)}(a)\right| \le \frac{n!\,M}{R^n} \qquad \text{Cauchy estimate}$$

Consequences: Liouville Theorem

An immediate consequence of the Cauchy estimates is the classical

Theorem (Liouville)

Every bounded entire function is constant

Proof:

If $|f(z)| < M$ for all z then

$$\sum_{n=0}^{+\infty} \left|c_n\right|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \left| f\left(a + re^{j\theta}\right)\right|^2 \mathrm{d}\theta \rightarrow \sum_{n=0}^{+\infty} \left|c_n\right|^2 r^{2n} < M^2$$

This is possible for all r only if $c_n = 0$ for all $n \geq 1$.

Exercise:

Show that Liouville Theorem implies that every polynomial with complex coefficients has at least one complex root (the *Fundamental Theorem of Algebra*)

Consequences: The Maximum Modulus Theorem

From

$$f\left(a\right) = c_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f\left(a + re^{j\theta}\right)\mathrm{d}\theta$$

one readily derives (try!) the following classical

Maximum Modulus Theorem

If Ω is a nonempty connected open set, $f \in H\left(\Omega\right)$, and $\bar{D}\left(a;r\right) \subset \Omega$. Then

$$\left|f\left(a\right)\right| \leq \max_{\theta}\left|f\left(a + re^{\theta}\right)\right|$$

equality occurring only if f is constant in Ω.

Consequently, $|f|$ **has no local maximum** at any point of Ω unless f is constant.

Applying the same reasoning to the real and imaginary parts of f, one finds that the same conclusion also holds for an arbitrary **harmonic function**.

The Global Cauchy Theorem and the Calculus of Residues

The Global Cauchy Theorem

Let us now **remove the restriction to convex regions** that was imposed in the local version of Cauchy Theorem.

To this aim, let a cycle Γ be the union of closed curves: $\Gamma = \gamma_1 \cup \ldots \cup \gamma_n$

Global Cauchy Theorem

Suppose $f \in H(\Omega)$, where Ω is an arbitrary open set in the complex plane. If Γ is a cycle in Ω that satisfies $\mathrm{Ind}_\Gamma(\alpha) = 0$ for every α not in Ω, then

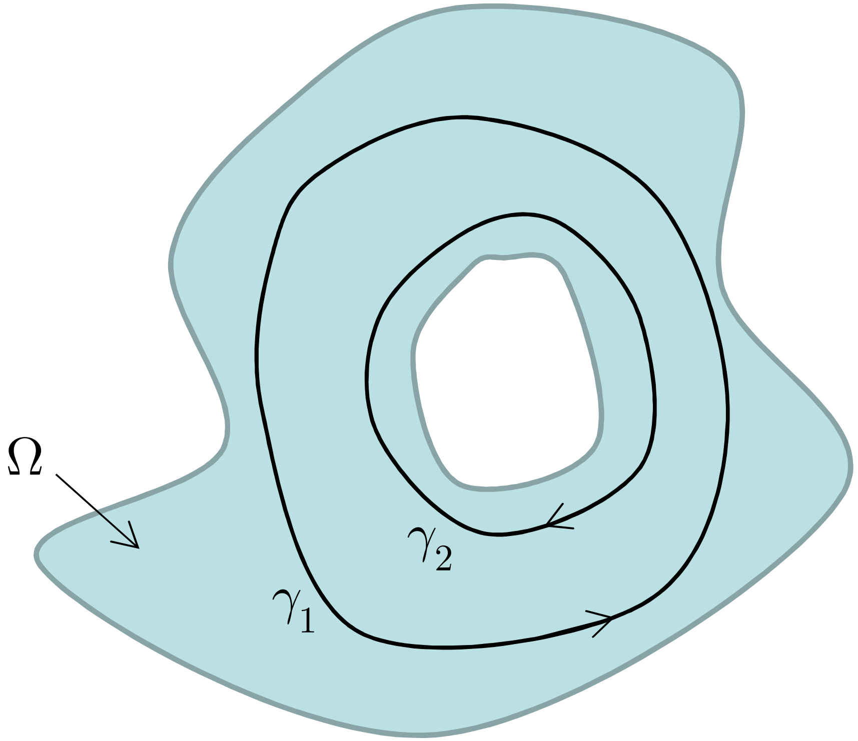
$$f(z) \cdot \mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi j} \int_\Gamma \frac{f(\xi)}{\xi - z} \mathrm{d}\xi, \quad (z \in \Omega \setminus \Gamma) \quad \text{and} \quad \int_\Gamma f(z) \mathrm{d}z = 0 \quad .$$

If Γ_0 and Γ_1 are cycles in Ω such that $\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$ for every α not in Ω, then

$$\int_{\Gamma_0} f(z) \mathrm{d}z = \int_{\Gamma_1} f(z) \mathrm{d}z$$

The Global Cauchy Theorem

Example



$$f \in H(\Omega)$$

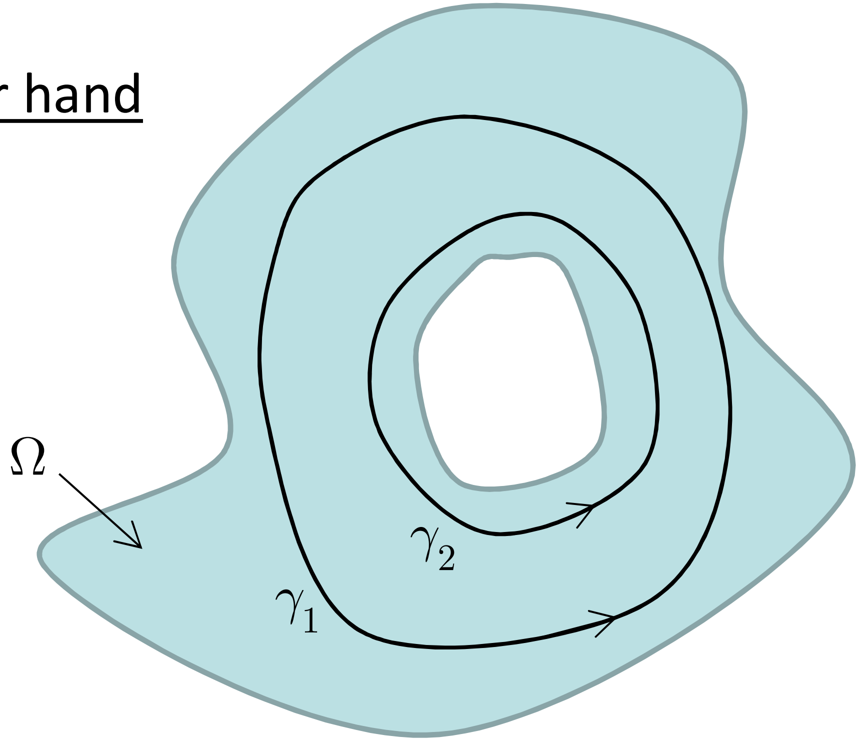
$$\downarrow$$

$$\int_{\Gamma} f(z)\,\mathrm{d}z = 0$$

Here the Global Cauchy Theorem **cannot** be applied to the closed paths γ_1 or γ_2, but **it can** be applied to the cycle $\Gamma = \gamma_1 \cup \gamma_2$ as it does not wind around any point in the complement of Ω.

The Global Cauchy Theorem

<u>On the other hand</u>



$$f \in H(\Omega)$$

$$\downarrow$$

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z$$

The Global Cauchy Theorem shows under what circumstances integration over a closed path **can be replaced** by integration over another, **without changing** the value of the integral.

In this connection, note that it can be shown that if γ_1 and γ_2 can be continuously deformed one to another remaining within Ω, then $\mathrm{Ind}_{\gamma_1}(\alpha) = \mathrm{Ind}_{\gamma_2}(\alpha)$ for every α not in Ω.

Applications: the Residue Theorem

A function f is said to be **meromorphic** in an open set Ω if there is a set $A \subset \Omega$ such that

a) A has no limit point in Ω.

b) $f \in H\left(\Omega \setminus A\right)$

c) f has a pole at each point of A, with principal part $\sum_{k=1}^{m} c_k \left(z-a\right)^{-k}$

The number c_1 is called the **residue** of f at a. We write: $c_1 = \mathrm{Res}(f;a)$

Residue Theorem

Suppose f is a meromorphic function in Ω and let A be the set of its poles. If Γ is a cycle in $\Omega \setminus \mathrm{A}$ such that $\mathrm{Ind}_{\Gamma}\left(\alpha\right) = 0$ for all α not in Ω . Then

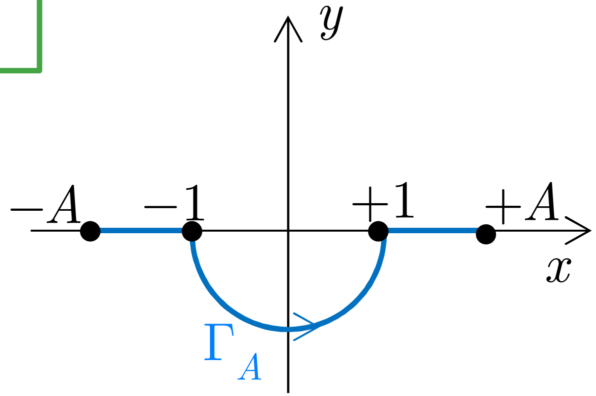
$$\frac{1}{2\pi j}\int_{\Gamma} f\left(z\right)\mathrm{d}z = \sum_{a\in A}\mathrm{Res}\left(f;a\right)\cdot \mathrm{Ind}_{\Gamma}\left(a\right)$$

Applications: Evaluation of Definite Integrals

The theory of complex functions allows for evaluating a number of definite integrals which otherwise could not be calculated.

First example:

Evaluate the real improper integral $\lim\limits_{A\to+\infty}\int_{-A}^{+A}\frac{\sin x}{x}e^{jxt}\mathrm{d}x$



Since $\frac{\sin z}{z}e^{jzt}$ is entire,

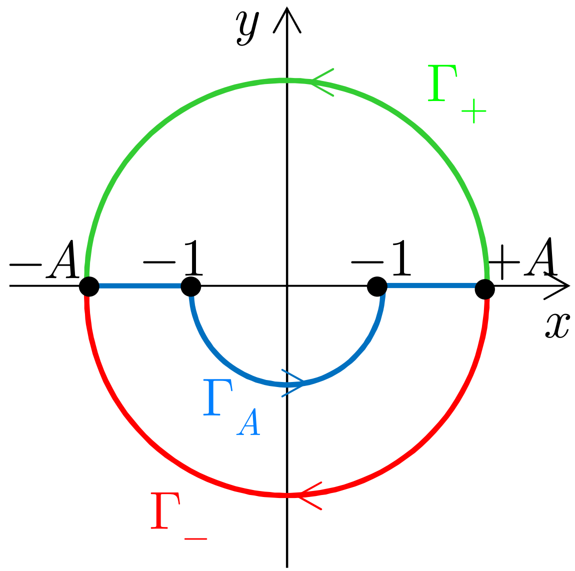
$$I_A=\int_{-A}^{+A}\frac{\sin x}{x}e^{jxt}\mathrm{d}x=\int_{\Gamma_A}\frac{\sin z}{z}e^{jzt}\mathrm{d}z$$

Now $2j\sin z=e^{jz}-e^{-jz}$, hence $I_A=\varphi_A(t+1)-\varphi_A(t-1)$ where

$$\frac{1}{\pi}\varphi_A(s)=\frac{1}{2\pi j}\int_{\Gamma_A}\frac{e^{jsz}}{z}\mathrm{d}z$$

Applications: Evaluation of Definite Integrals

We complete Γ_A to a closed path in two ways, using Γ_- or Γ_+ :



The function e^{jsz} / z has a simple pole at $z=0$ with residue 1, hence

$$\frac{1}{\pi}\varphi_A(s) = \frac{1}{2\pi}\int_{-\pi}^{0} \exp\left(jsA\,e^{j\theta}\right)\mathrm{d}\theta \qquad \frac{1}{\pi}\varphi_A(s) = 1 - \frac{1}{2\pi}\int_{0}^{\pi} \exp\left(jsA\,e^{j\theta}\right)\mathrm{d}\theta$$

Applications: Evaluation of Definite Integrals

Note that $\left|\exp\left(jsA\,e^{j\theta}\right)\right| = \exp\left(-A\,s\sin\theta\right)$

so the integrals over Γ_- and Γ_+ tend to zero as A tends to infinity for $s<0$ and $s>0$, respectively.

In fact, for, e.g., Γ_+ one has: $$\left|\int_0^\pi \exp\left(jsA\,e^{j\theta}\right)\mathrm{d}\theta\right| \leq \int_0^\pi \left|\exp\left(jsA\,e^{j\theta}\right)\right|\mathrm{d}\theta = \int_0^\pi e^{-sA\sin\theta}\mathrm{d}\theta$$

$$= 2\int_0^{\pi/2} e^{-sA\sin\theta}\mathrm{d}\theta$$

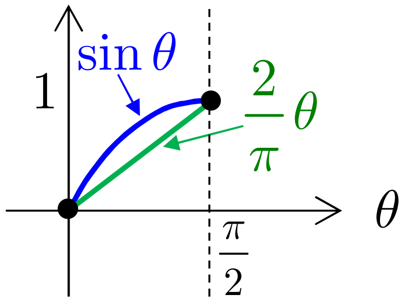
$$\leq 2\int_0^{\pi/2} e^{-sA2\theta/\pi}\mathrm{d}\theta$$

$$= \frac{\pi}{sA}\left(1-e^{-sA}\right)\underset{A\to+\infty}{\longrightarrow} 0$$

The $\sin(\theta)$ function is convex in $[0,\pi/2]$:

$$\sin\theta \geq \frac{2}{\pi}\theta$$

(Jordan’s inequality)



Applications: Evaluation of Definite Integrals

Therefore we find

$$\lim_{A\to+\infty} \varphi_A\left(s\right) = \begin{cases} \pi, & s>0 \\ 0, & s<0 \end{cases}$$

and finally, remembering that $I_A = \varphi_A\left(t+1\right) - \varphi_A\left(t-1\right)$

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x} e^{jxt}\,\mathrm{d}x = \lim_{A\to+\infty} I_A = \begin{cases} \pi, & -1<t<+1 \\ 0, & \left|t\right|>1 \end{cases} = \pi\,\mathrm{rect}_1\left(t\right)$$

Applications: Evaluation of Definite Integrals

Second example:

Evaluate the real improper integrals

$$I_{\mathrm{C}} = \lim_{A \to +\infty} \int_0^{+A} \cos t^2 \mathrm{d}t \qquad I_{\mathrm{S}} = \lim_{A \to +\infty} \int_0^{+A} \sin t^2 \mathrm{d}t$$

These are the limit values of the real Fresnel integrals

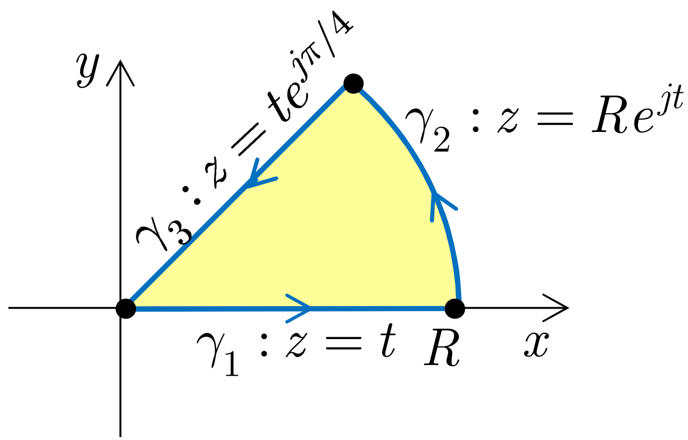
$$C(x) = \int_0^{x} \cos t^2 \mathrm{d}t \qquad S(x) = \int_0^{x} \sin t^2 \mathrm{d}t$$

as x tends to infinity.

Here we make use of the contour integral of the function

$$e^{-z^2}$$

around the boundary of a sector-shaped region of the first quadrant of the complex plane:



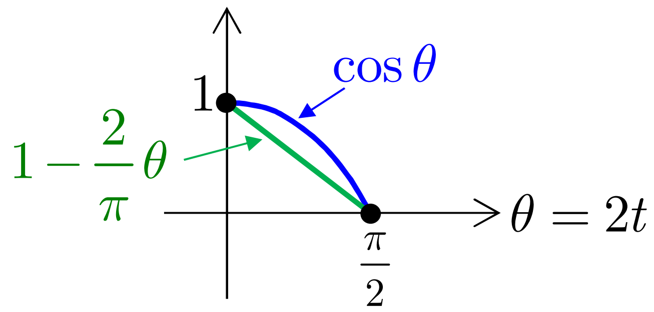
Applications: Evaluation of Definite Integrals

The integral along the circular arc tends to zero as R tends to infinity. In fact,

$$\left|\int_{\gamma_2} e^{-z^2}\,\mathrm{d}z\right| \le \int_{\gamma_2}\left|e^{-z^2}\right|\mathrm{d}z = R\int_0^{\pi/4} e^{-R^2\cos 2t}\,\mathrm{d}t$$

But the $\cos(\theta)$ function is convex in $[0,\pi/2]$:

$$\cos 2t \ge 1-\frac{2}{\pi}2t$$



(Jordan's inequality) hence

$$\left|\int_{\gamma_2} e^{-z^2}\,\mathrm{d}z\right| \le R\int_0^{\pi/4} e^{-R^2\left(1-\frac{4}{\pi}t\right)}\mathrm{d}t = \frac{\pi}{4R}\left(1-e^{-R^2}\right)\underset{R\to+\infty}{\longrightarrow} 0$$

Applications: Evaluation of Definite Integrals

By Cauchy Theorem we thus have

$$\int_{\gamma_1} e^{-z^2} \mathrm{d}z + \int_{\gamma_3} e^{-z^2} \mathrm{d}z = 0$$

Now,

$$\int_{\gamma_1} e^{-z^2} \mathrm{d}z = \int_0^R e^{-t^2} \mathrm{d}t \underset{R \to +\infty}{\longrightarrow} \int_0^{+\infty} e^{-t^2} \mathrm{d}t = \frac{\sqrt{\pi}}{2}$$

(Gauss integral)

Whereas

$$\int_{\gamma_3} e^{-z^2} \mathrm{d}z = -\int_0^R e^{-\left(te^{j\pi/4}\right)^2} \mathrm{d}\left(te^{j\pi/4}\right) = -\frac{1+j}{\sqrt{2}} \int_0^R e^{-jt^2} \mathrm{d}t \underset{R \to +\infty}{\longrightarrow} -\frac{1+j}{\sqrt{2}}\left(I_{\mathrm{C}} - jI_{\mathrm{S}}\right)$$

Applications: Evaluation of Definite Integrals

So Cauchy theorem gives

$$\frac{\sqrt{\pi}}{2} - \frac{1+j}{\sqrt{2}}\left(I_{\mathrm{C}} - jI_{\mathrm{S}}\right) = 0$$

By separating the real and imaginary parts:

$$I_{\mathrm{C}} + I_{\mathrm{S}} = \sqrt{\frac{\pi}{2}}$$
$$I_{\mathrm{C}} - I_{\mathrm{S}} = 0$$

$$\Rightarrow \quad \boxed{I_{\mathrm{C}} = I_{\mathrm{S}} = \frac{1}{2}\sqrt{\frac{\pi}{2}} = \sqrt{\frac{\pi}{8}}}$$

References

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