MULTIPLE NON PARAMETRIC TESTS II (DA_2022)

Andrea Giansanti
Dipartimento di Fisica, Sapienza Università di Roma

Andrea.Giansanti@roma1.infn.it

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Dipartimento di Fisica

Sapienza
Università di Roma

Outline L 21

- F test statistics (see Rosners' chapter 12 see R chap. 8.6)
- Graded Homework Formally derive equation 12.4 and equation 12.5 in Rosner's textbook
- A glance to the jungle…(Rosner's road map)
- Further discussion of the ANOVA one way test
- Kruskal-Wallis test

…A GLIMPSE TO THE JUNGLE

FIGURE 11.33 Flowchart for appropriate methods of statistical inference

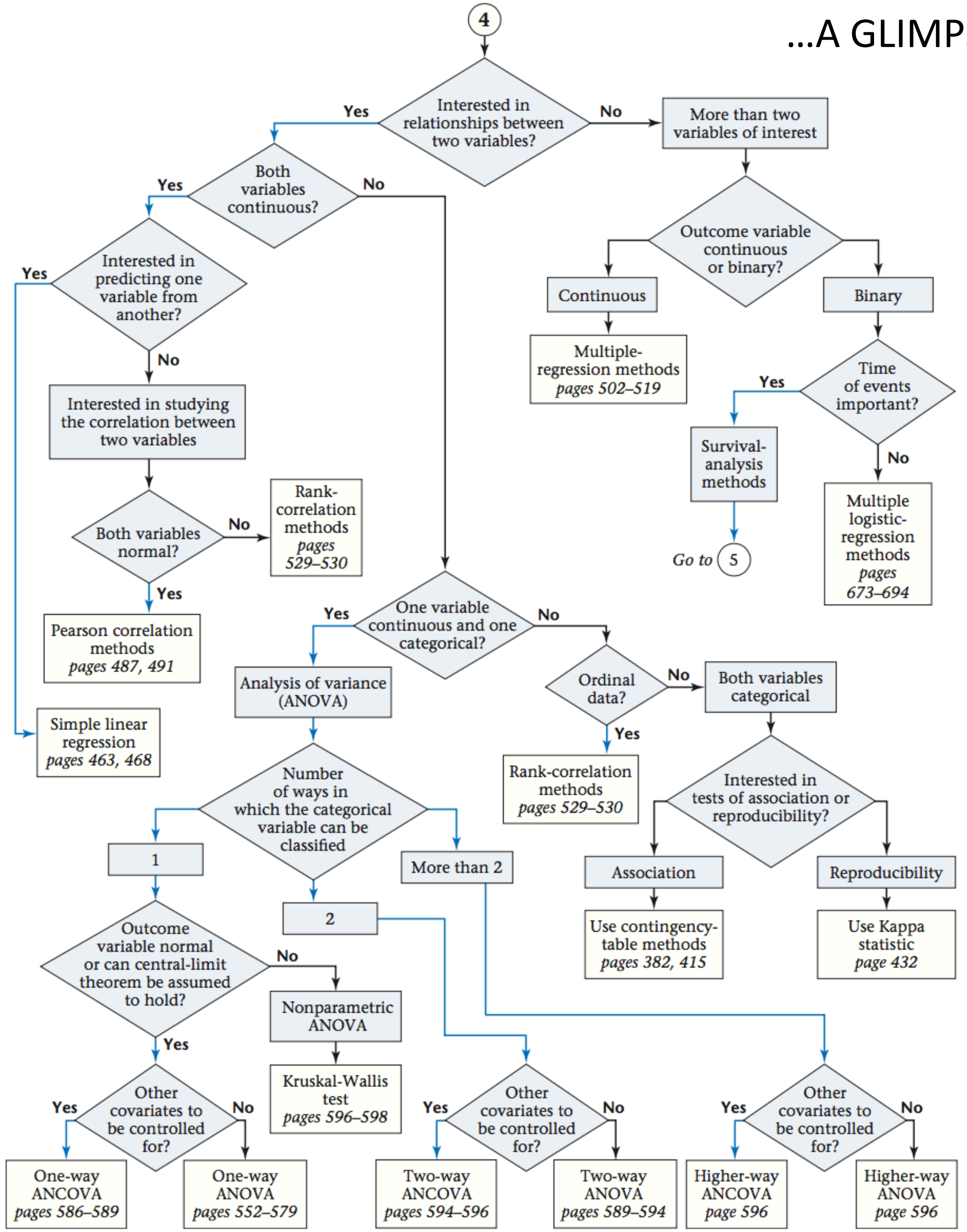


TABLE 3 A comparison of methods to test differences between group means according to whether the tests assume normal distributions. (Red numbers in parentheses refer to the chapter that discusses the test.)

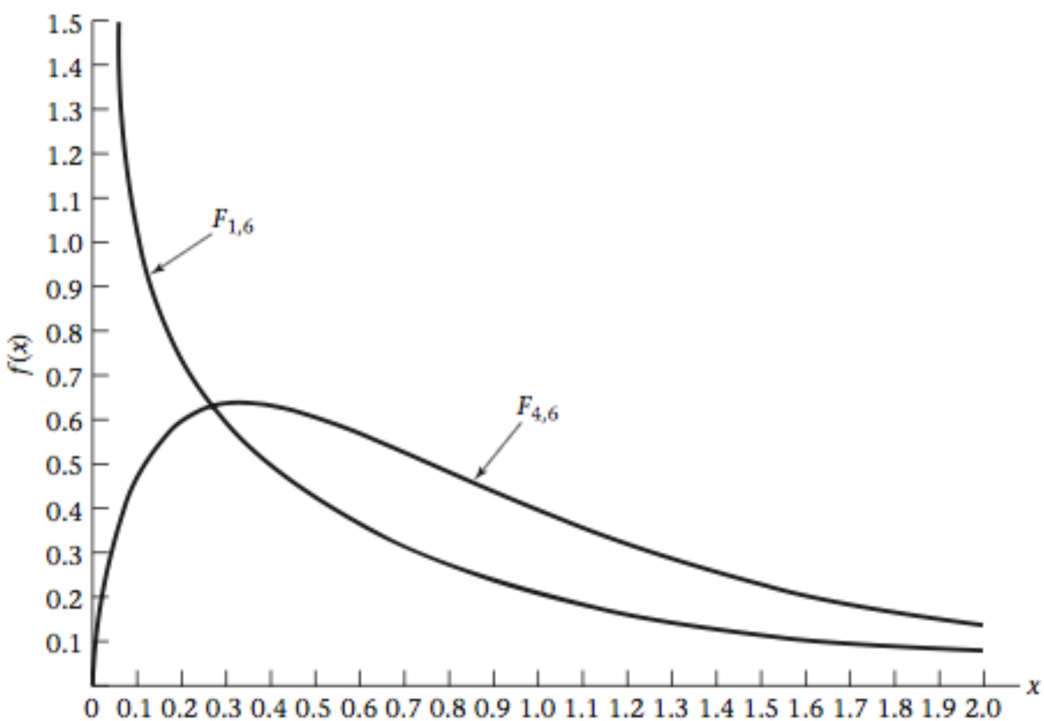
Number of treatments	Tests assuming normal distribution	Tests not assuming normal distributions
Two treatments (independent samples)	Two-sample *t*-test (12) Welch's *t*-test (used when variance is unequal in the two groups) (12)	Mann-Whitney *U*-test (13)
Two treatments (paired data)	Paired *t*-test (12)	Sign test (13)
More than two treatments	ANOVA (15)	Kruskal-Wallis test (15)

The *F* Distribution

The distribution of the variance ratio $\left(S_1^2/S_2^2\right)$ was studied by statisticians R. A. Fisher and G. Snedecor. It can be shown that the variance ratio follows an ***F* distribution** under the null hypothesis that $\sigma_1^2 = \sigma_2^2$. There is no unique *F* distribution but instead a family of *F* distributions. This family is indexed by two parameters termed the *numerator* and *denominator degrees of freedom*, respectively. If the sizes of the first and second samples are n_1 and n_2 respectively, then the variance ratio follows an *F* distribution with $n_1 - 1$ (numerator *df*) and $n_2 - 1$ (denominator *df*), which is called an F_{n_1-1,n_2-1} distribution.

The *F* distribution is generally positively skewed, with the skewness dependent on the relative magnitudes of the two degrees of freedom. If the numerator *df* is 1 or 2, then the distribution has a mode at 0; otherwise, it has a mode greater than 0. The distribution is illustrated in Figure 8.5. Table 8 in the Appendix gives the percentiles of the *F* distribution for selected values of the numerator and denominator *df*.

Probability density for the *F* distribution



The **100 × *p*th percentile of an *F* distribution** with d_1 and d_2 degrees of freedom is denoted by $F_{d_1,d_2,p}$. Thus,

$$Pr\left(F_{d_1,d_2} \le F_{d_1,d_2,p}\right) = p$$

The *F* table is organized such that the numerator *df* (d_1) is shown in the first row, the denominator *df* (d_2) is shown in the first column, and the various percentiles (p) are shown in the second column.

Find the upper first percentile of an *F* distribution with 5 and 9 *df*.

Solution: $F_{5,9,.99}$ must be found. Look in the 5 column, the 9 row, and the subrow marked .99 to obtain

$$F_{5,9,.99} = 6.06$$

TABLE 8 Percentage points of the *F* distribution ($F_{d_1,d_2,p}$)

df for denominator, d_2	*p*	*df* for numerator, d_1: 1	2	3	4	5	6	7	8	12	24	∞
1	.90	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	60.71	62.00	63.33
	.95	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	243.9	249.1	254.3
	.975	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	976.7	997.2	1018.
	.99	4052.	5000.	5403.	5625.	5764.	5859.	5928.	5981.	6106.	6235.	6366.
	.995	16211.	20000.	21615.	22500.	23056.	23437.	23715.	23925.	24426.	24940.	25464.
	.999	405280.	500000.	540380.	562500.	576400.	585940.	592870.	598140.	610670.	623500.	636620.
2	.90	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.41	9.45	9.49
	.95	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.41	19.45	19.50
	.975	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.42	39.46	39.50
	.99	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.42	99.46	99.50
	.995	198.5	199.0	199.2	199.2	199.3	199.3	199.4	199.4	199.4	199.5	199.5
	.999	998.5	999.0	999.2	999.2	999.3	999.3	999.4	999.4	999.4	999.5	999.5
3	.90	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.22	5.18	5.13
	.95	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.74	8.64	8.53
	.975	17.44	16.04	15.44	15.10	14.88	14.74	14.62	14.54	14.34	14.12	13.90
	.99	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.05	26.60	26.13
	.995	55.55	49.80	47.47	46.20	45.39	44.84	44.43	44.13	43.39	42.62	41.83
	.999	167.00	148.5	141.1	137.1	134.6	132.8	131.6	130.6	128.3	125.9	123.5
4	.90	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.90	3.83	3.76
	.95	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	5.91	5.77	5.63
	.975	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.75	8.51	8.26
	.99	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.37	13.93	13.46
	.995	31.33	26.28	24.26	23.16	22.46	21.98	21.62	21.35	20.70	20.03	19.32
	.999	74.14	61.25	56.18	53.44	51.71	50.53	49.66	49.00	47.41	45.77	44.05
5	.90	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.27	3.19	3.10
	.95	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.68	4.53	4.36
	.975	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.52	6.28	6.02
	.99	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	9.89	9.47	9.02
	.995	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.38	12.78	12.14
	.999	47.18	37.12	33.20	31.09	29.75	28.83	28.16	27.65	26.42	25.13	23.79
6	.90	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.90	2.82	2.72
	.95	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.00	3.84	3.67
	.975	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.37	5.12	4.85
	.99	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.72	7.31	6.88
	.995	18.64	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.03	9.47	8.88
	.999	35.51	27.00	23.70	21.92	20.80	20.03	19.46	19.03	17.99	16.90	15.75
7	.90	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.67	2.58	2.47
	.95	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.57	3.41	3.23
	.975	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.67	4.42	4.14
	.99	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.47	6.07	5.65
	.995	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.18	7.65	7.08
	.999	29.25	21.69	18.77	17.20	16.21	15.52	15.02	14.63	13.71	12.73	11.70
8	.90	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.50	2.40	2.29
	.95	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.28	3.12	2.93
	.975	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.20	3.95	3.67
	.99	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.67	5.28	4.86
	.995	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.01	6.50	5.95
	.999	25.42	18.49	15.83	14.39	13.49	12.86	12.40	12.04	11.19	10.30	9.33
9	.90	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.38	2.28	2.16
	.95	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.07	2.90	2.71
	.975	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	3.87	3.61	3.33
	.99	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.11	4.73	4.31
	.995	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.23	5.73	5.19
	.999	22.86	16.39	13.90	12.56	11.71	11.13	10.70	10.37	9.57	8.72	7.81

Generally, F distribution tables give only upper percentage points because the symmetry properties of the F distribution make it possible to derive the lower percentage points of any F distribution from the corresponding upper percentage points of an F distribution with the degrees of freedom reversed. Specifically, note that under H_0, S_2^2/S_1^2 follows an F_{d_2,d_1} distribution. Therefore,

$$Pr\left(S_2^2/S_1^2 \geq F_{d_2,d_1,1-p}\right) = p$$

By taking the inverse of each side and reversing the direction of the inequality, we get

$$Pr\left(\frac{S_1^2}{S_2^2} \leq \frac{1}{F_{d_2,d_1,1-p}}\right) = p$$

Under H_0, however, S_1^2/S_2^2 follows an F_{d_1,d_2} distribution. Therefore,

$$Pr\left(\frac{S_1^2}{S_2^2} \leq F_{d_1,d_2,p}\right) = p$$

It follows from the last two inequalities that

$$F_{d_1,d_2,p} = \frac{1}{F_{d_2,d_1,1-p}}$$

This principle is summarized as follows.

Computation of the Lower Percentiles of an F Distribution

The **lower pth percentile** of an F distribution with d_1 and d_2 *df* is the reciprocal of the **upper pth percentile** of an F distribution with d_2 and d_1 *df*. In symbols,

$$F_{d_1,d_2,p} = 1/F_{d_2,d_1,1-p}$$

Thus, from Equation 8.14 we see that the lower pth percentile of an F distribution is the same as the inverse of the upper pth percentile of an F distribution with the degrees of freedom reversed.

Overall *F* Test for One-Way ANOVA

To test the hypothesis H_0: $\alpha_i = 0$ for all i vs. H_1: at least one $\alpha_i \neq 0$, use the following procedure:

(1) Compute the Between SS, Between MS, Within SS, and Within MS using Equation 12.5 and Definitions 12.5 and 12.6.

(2) Compute the test statistic F = Between MS/Within MS, which follows an F distribution with $k - 1$ and $n - k$ *df* under H_0.

(3) If $F > F_{k-1,n-k,1-\alpha}$ then reject H_0
If $F \leq F_{k-1,n-k,1-\alpha}$ then accept H_0

(4) The exact *p*-value is given by the area to the right of F under an $F_{k-1,n-k}$ distribution $= Pr(F_{k-1,n-k} > F)$.

The acceptance and rejection regions for this test are shown in Figure 12.2. Computation of the exact *p*-value is illustrated in Figure 12.3. The results from the ANOVA are typically displayed in an ANOVA table, as in Table 12.2.

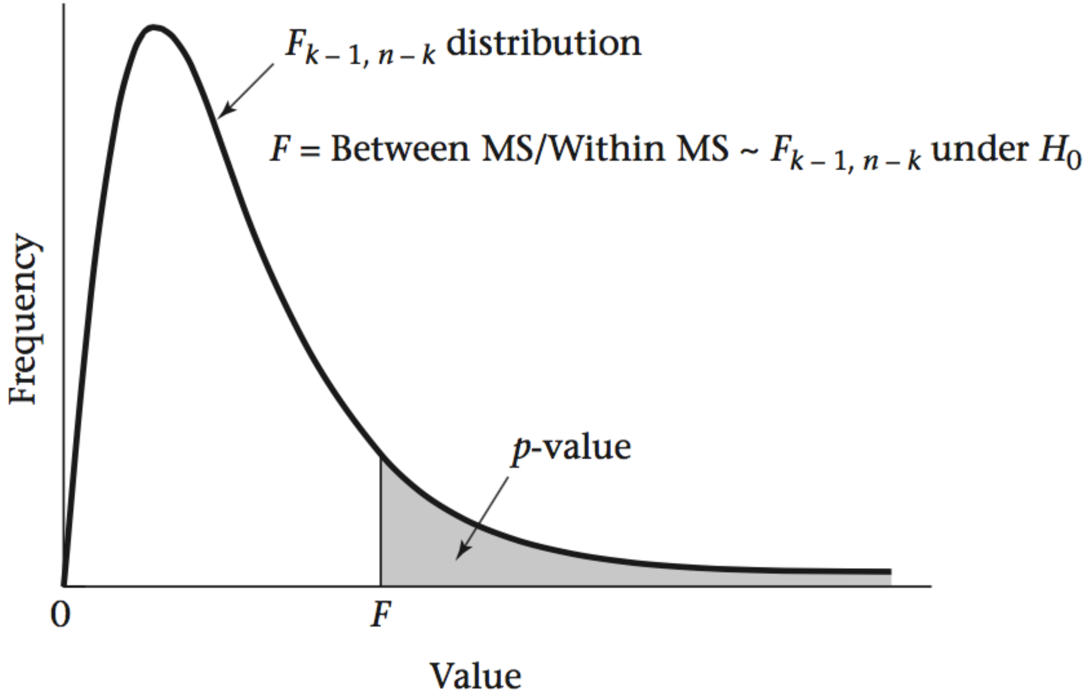
Display of one-way ANOVA results

Source of variation	SS	*df*	MS	*F* statistic	*p*-value
Between	$\sum_{i=1}^{k} n_i \bar{y}_i^2 - \frac{y_{..}^2}{n} = B$	$k-1$	$\frac{B}{k-1}$	$\frac{B/(k-1)}{A/(n-k)} = F$	$Pr(F_{k-1,n-k} > F)$
Within	$\sum_{i=1}^{k} (n_i - 1)s_i^2 = A$	$n-k$	$\frac{A}{n-k}$		
Total	Between SS + Within SS				

ANOVA table for FEF data in Table 12.1

	SS	*df*	MS	*F* statistic	*p*-value
Between	184.38	5	36.875	58.0	$p < .001$
Within	663.87	1044	0.636		
Total	848.25				

Computation of the exact *p*-value for the overall *F* test for one-way ANOVA



Refer to Table 8 in the Appendix and find that

$$F_{5,120,.999} = 4.42$$

CONCLUSION

Because $F_{5,1044,.999} < F_{5,120,.999} = 4.42 < 58.0 = F$

it follows that $p < .001$. The exact *p*-value obtained from Stata = *F*tail(5,1044,58) = 2.5×10^{-53}. Therefore, we can reject H_0, that all the means are equal, and can conclude that at least two of the means are significantly different. These results are displayed in an ANOVA table (Table 12.3).

THE KRUSKAL-WALLIS TEST

In some instances we want to compare means among more than two samples, but either the underlying distribution is far from being normal or we have ordinal data. In these situations, a nonparametric alternative to the one-way ANOVA described in Sections 12.1–12.4 of this chapter must be used.

THE KW TEST IS BASED ON THE CHI_SQUARE STATISTICS

$$H = \frac{12}{N(N+1)} \sum_{i=1}^{k} n_i \left(\overline{R}_i - \overline{\overline{R}} \right)^2$$

where $\overline{R}_i$ = average rank in the ith sample and $\overline{\overline{R}}$ = average rank over all samples combined. Thus, if the average rank is about the same in all samples, then $\left|\overline{R}_i - \overline{\overline{R}}\right|$ will tend to be small and H_0 will be accepted. On the contrary, if the average rank is very different across samples, then $\left|\overline{R}_i - \overline{\overline{R}}\right|$ will tend to be large and H_0 will be rejected.

The Kruskal-Wallis Test

To compare the means of k samples ($k > 2$) using nonparametric methods, use the following procedure:

(1) Pool the observations over all samples, thus constructing a combined sample of size $N = \Sigma n_i$

(2) Assign ranks to the individual observations, using the average rank in the case of tied observations.

(3) Compute the rank sum R_i for each of the k samples.

(4) If there are no ties, compute the test statistic

$$H = H^* = \frac{12}{N(N+1)} \times \sum_{i=1}^{k} \frac{R_i^2}{n_i} - 3(N+1)$$

If there are ties, compute the test statistic

$$H = \frac{H^*}{1 - \frac{\sum_{j=1}^{g}\left(t_j^3 - t_j\right)}{N^3 - N}}$$

where t_j refers to the number of observations (i.e., the frequency) with the same value in the jth cluster of tied observations and g is the number of tied groups.

(5) For a level α test,

if $H > \chi^2_{k-1,1-\alpha}$ then reject H_0

if $H \le \chi^2_{k-1,1-\alpha}$ then accept H_0

(6) To assess statistical significance, the p-value is given by

$$p = Pr\left(\chi^2_{k-1} > H\right)$$

(7) This test procedure should be used only if minimum $n_i \ge 5$ (i.e., if the smallest sample size for an individual group is at least 5).

The acceptance and rejection regions for this test are shown in Figure 12.12. Computation of the exact p-value is given in Figure 12.13.

Multiple Comparisons—Bonferroni Approach

In many studies, comparisons of interest are specified before looking at the actual data, in which case the *t* test procedure in Equation 12.12 and the linear-contrast procedure in Equation 12.13 are appropriate. In other instances, comparisons of interest are only specified after looking at the data. In this case a large number of potential comparisons are often possible. Specifically, if there are a large number of groups and every pair of groups is compared using the *t* test procedure in Equation 12.12, then some significant differences are likely to be found just by chance.

Suppose there are 10 groups. Thus, there are $\binom{10}{2} = 45$ possible pairs of groups to be compared. Using a 5% level of significance would imply that .05(45), or about two comparisons, are likely to be significant by chance alone. How can we protect ourselves against the detection of falsely significant differences resulting from making too many comparisons?

Several procedures, referred to as **multiple-comparisons procedures,** ensure that too many falsely significant differences are not declared. The basic idea of these procedures is to ensure that the *overall probability of declaring any significant differences between all possible pairs of groups* is maintained at some fixed significance level (say α). One of the simplest and most widely used such procedures is the method of *Bonferroni adjustment*. This method is summarized as follows.

Comparison of Pairs of Groups in One-Way ANOVA—Bonferroni Multiple-Comparisons Procedure

Suppose we wish to compare two specific groups, arbitrarily labeled as group 1 and group 2, among k groups. To test the hypothesis H_0: $\alpha_1 = \alpha_2$ vs. H_1: $\alpha_1 \neq \alpha_2$, use the following procedure:

(1) Compute the pooled estimate of the variance s^2 = Within MS from the one-way ANOVA.

(2) Compute the test statistic

$$t = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

(3) For a two-sided level α test, let $\alpha^* = \alpha \Big/ \binom{k}{2}$

If $t > t_{n-k,1-\alpha^*/2}$ or $t < t_{n-k,\alpha^*/2}$ then reject H_0

If $t_{n-k,\alpha^*/2} \le t \le t_{n-k,1-\alpha^*/2}$ then accept H_0

The acceptance and rejection regions for this test are given in Figure 12.7.

(4) The Bonferroni corrected p-value $= \min\left[2\binom{k}{2}\Pr(t_{n-k} > |t|), 1\right]$

$= \min[k(k-1)\Pr(t_{n-k} > |t|, 1)] = \binom{k}{2}$ *LSD p*-value.

This test is called the *Bonferroni multiple-comparisons procedure*.

The rationale behind this procedure is that in a study with k groups, there are $\binom{k}{2}$ possible two-group comparisons. Suppose each two-group comparison is conducted at the α^* level of significance. Let E be the event that at least one of the two-group comparisons is statistically significant. $Pr(E)$ is sometimes referred to as the "experiment-wise type I error." We wish to determine the value α^* such that $Pr(E) = \alpha$. To find α^*, we note that

$Pr(\bar{E}) = Pr$(none of the two-group comparisons are statistically significant) = $1 - \alpha$. If each of the two-group comparisons were independent, then from the multiplication law of probability, $Pr(\bar{E}) = (1 - \alpha^*)^c$, where $c = \binom{k}{2}$. Therefore,

$$1 - \alpha = (1 - \alpha^*)^c$$

If α^* is small, then it can be shown that the right-hand side of Equation 12.15 can be approximated by $1 - c\alpha^*$. Thus

$$1 - \alpha \cong 1 - c\alpha^*$$

or

$$\alpha^* \cong \alpha/c = \alpha \bigg/ \binom{k}{2} \text{ as given in Equation 12.14.}$$

Usually all the two-group comparisons are *not* statistically independent, whereby the appropriate value α^* is greater than $\alpha \big/ \binom{k}{2}$. Thus, the Bonferroni procedure is conservative in the sense that $Pr(E) < \alpha$.