# Luciano Maiani:
# Lezione Fermi 11
# Quantum Electro Dynamics, QED. Renormalization

1. Divergences in classical ED, the UV cutoff
2. Renormalization!
3. The electron anomaly
4. The muon anomaly
5. Conclusions
6. Epilogue

# Divergences in classical ED

- Inertia of an electron=mass + energy stored in the electric field = $= m_0 + E_{field}$
- for a uniform charge distribution within a sphere of radius R

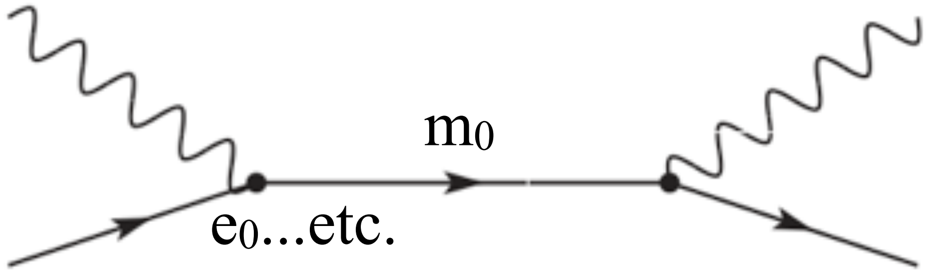
$$E_{field} = \frac{e^2}{R}$$

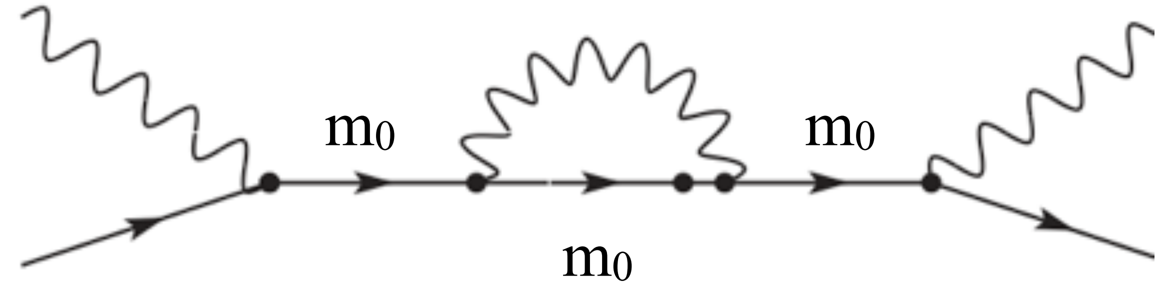
- it is infinite for a point-like particle.

# cutoff

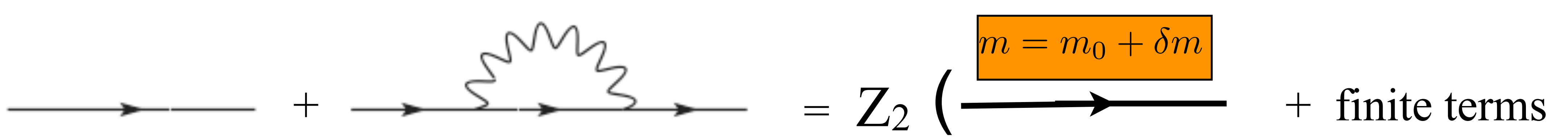
- Divergences in QED arise because we have to add amplitudes A(q) for all values of q, up to infinity, and some amplitudes do not converge to zero fast enough
- we may cut the integration at some value of $q=q_{max}=\Lambda$ and obtain a finite result which depends from $\Lambda$ (and diverges as $\Lambda\rightarrow+\infty$)
- what did we gain:
  - for e sufficiently small, the contributions at fixed $\Lambda$ can be treated as small perturbations
  - we can discuss structure and interpretation of the potentially divergent terms.
- Another way of introducing a cutoff is to *discretize space-time* in a lattice of points, say a cubic lattice:
- $x_1$, $x_2$, ....spaced by ***a*** (lattice spacing) in a large, fixed, volume V
- this is particularly suited for numerical calculations (lattice gauge theories): we have now a finite number of points and a finite number of variables at each point, which can be put in a (large) computer memory
- wave propagates on the lattice, but only up to a minimal wave lenght $\lambda$=a, i.e. momentum $q_{max} = \Lambda = \hbar a^{-1}$
- divergent amplitudes computed on the lattice, diverge for $a\rightarrow 0$.
- The classical result for the electron mass would be described, in this language, as a result diverging linearly with $\Lambda$ or $a^{-1}$: $m = m_0 + \delta m = m_0 + \alpha\Lambda$
- for $\Lambda$ fixed, this is a small correction if $\alpha$ is sufficiently small.

# 2 Renormalization




- let us indicate with $m_0$ the electron mass ***in absence of the electromagnetic interaction***, like we did for the classical case
  - this is the mass that appears in the action
  - we have eplicitly indicated in the diagrams above that $m_0$ appears in the amplitudes for the propagation of the electron between the vertices above
- similarly we indicate with $e_0$ the coupling ***in absence of the electromagnetic interaction,*** which is what appears everywhere in the amplitudes for the vertices above
- the amplitude for the propagation of the electron is simply the sum of what is between the external vertices, and it can be shown that this is equal to
  - the amplitude for the propagation without interaction, but with $m_0 \to m_0 + \delta m$,
  - plus corrections which remain finite for $\Lambda \to \infty$
  - and factors which can be absorbed in a rescaling of the charge of the contiguous vertices



# charge renormalization

- in a similar way, corrections of vertices and of the photon propagation amplitude can be absorbed in the definition of a new electric charge:

$$e = \frac{Z_2}{Z_1}\sqrt{Z_3}e_0$$

- .. and finite corrections
- note that $m_0$ and $e_0$ are not measurable in principle, only m and e are.

***Conclusion***

if we make the limit $\Lambda \to +\infty$ by ***keeping fixed the physical mass and charge,*** we obtain an amplitude which is equal to the lowest order result ***plus finite corrections of higher order in α***

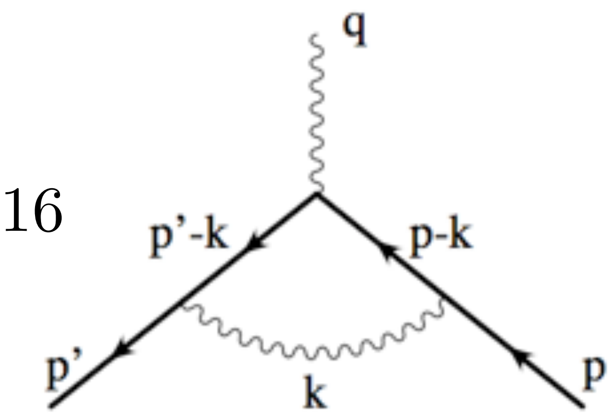
In other words

***The perturbative expansion expressed in terms of the physical mass and the physical charge is made of finite terms***

***these finite terms are responsible, among other things, of the electron (and muon) anomaly, of the Lamb shift and of many others effects, which therefore can be computed and compared to the experimental data***

***A much harder and still open problem: will the perturbative expansion converge itself to a finite result ?***

# 3. The electron anomaly




- The lowest order calculation by Schwinger: $a_e^{(2)} = \dfrac{\alpha}{2\pi} \approx 0.00116$ reproduces well the early result by Kusch.
- The present experimental determination, due to G. Gabrielse and collaborators (2008) has reached an extraordinary precision:

$$a_e(\mathrm{exp}) = 1\ 159\ 652\ 180.73\ (0.28) \times 10^{-12} \quad [0.24\ ppb]$$

Hanneke,Fogwell,Gabrielse, PRL 100, 120801 (2008)

- Theoretical calculations of the prediction of QED is a titanic enterprise, carried over several decades by a large number of investigators in several parts of the world.
- Here the picture a of three protagonists of the saga (Ettore Remiddi is professor at Bologna University)

- we can write:

what follows is taken from a recent talk by T. Knoshita

$a_e = a_e(\mathrm{QED}) + a_e(\mathrm{hadron}) + a_e(\mathrm{electroweak})$, where

$$a_e(\mathrm{QED}) = A_1 + A_2(m_e/m_\mu) + A_2(m_e/m_\tau) + A_3(m_e/m_\mu, m_e/m_\tau)$$

$$A_i = A_i^{(2)}\left(\frac{\alpha}{\pi}\right) + A_i^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + A_i^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + \ldots, i = 1,2,3$$

- First four $A_1$ terms are known analytically or by numerical integration

$A_1^{(2)} = \quad 0.5$ — 1 Feynman diagram (analytic)
$A_1^{(4)} = -0.328\,478\,965\ldots$ — 7 Feynman diagrams (analytic)
$A_1^{(6)} = \quad 1.181\,241\,456\ldots$ — 72 Feynman diagrams (analytic,numerical)

Laporta, Remiddi, PLB 379, 283 (1996)
Kinoshita, PRL 75, 4728 (1995)

$A_1^{(8)} = -1.914\,4\,(35)$ — 891 Feynman diagrams (numerical)

Kinoshita,Nio, PRD 73, 013003 (2006)
Aoyama,Hayakawa,Kinoshita,Nio, PRD 77, 053012 (2008)

- $A_2$ term is small but not negligible: $\sim 2.72 \times 10^{-12}$.
- $A_3$ term is completely negligible at present: ($\sim 2.4 \times 10^{-21}$).
- Hadronic and electroweak contributions (in SM) are also known
  a) $a_e(\mathrm{hadron}) = 1.689\,(20) \times 10^{-12}$

  Jegerlehner, priv. com. 1996
  Krause, PLB 390, 392 (1997)
  Nyfeller, arXiv:0901.1172 [hep-ph]

  b) $a_e(\mathrm{EW}) = 0.030 \times 10^{-12}$

  Czarnecki *et al.*, PRL 76, 3267 (1996)

- comparison of the QED prediction with the experimental value requires a very precise determination of α.
- Kinoshita takes the optical lattice determination, to find:

$$a_e(\mathrm{Rb}) = 1\;159\;652\;182.79\;(0.11)(0.37)(7.72) \times 10^{-12},$$

$$a_e(\exp) - a_e(\mathrm{Rb}) = -2.06\;(7.72) \times 10^{-12}.$$

where

$$\alpha^{-1}(\mathrm{Rb}) = 137.035\;998\;84\;(91). \quad [6.7\;ppb],$$

is the value obtained by an optical lattice method.

P. Cladé et al., PRA 74, 052109 (2006)

- and

$$a_e(\exp) = 1\;159\;652\;180.73\;(0.28) \times 10^{-12} \quad [0.24\;ppb]$$

Hanneke,Fogwell,Gabrielse, PRL 100, 120801 (2008)

- work is still in progress to improve the theoretical calculation of $a_e$ and to improve the experimental determination of α and $a_e$.
- Due to the very small value of the electron mass, $a_e$ is very unsensitive, even to this precision, to hadronic effects, to the extension of QED into the full Electroweak theory and to further extensions of the Standard Model at high energy (if any).

# 4. The muon anomaly

Brookhaven National Lab
Muon g-2 Collaboration
(Y. Semertzidis, ICHEP 2002)

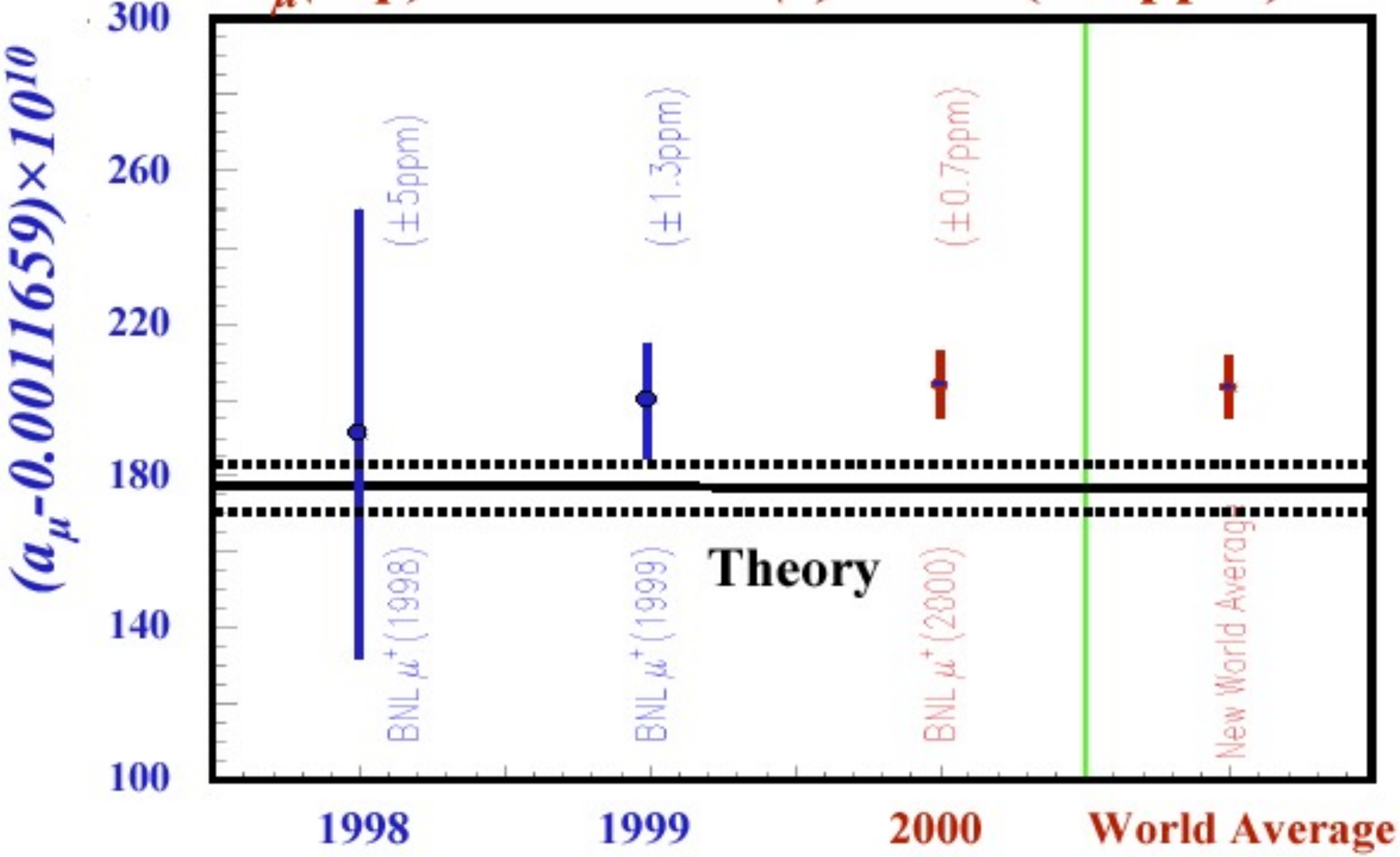
Results

From the Data of 2000:

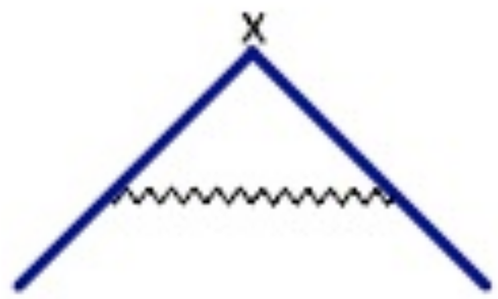
$a_\mu(exp)=11\ 659\ 204(7)(5)\times 10^{-10}$ (0.7 ppm)

Exp. World Average:

$a_\mu(exp)=11\ 659\ 203(8)\times 10^{-10}$ (0.7 ppm)



# Pure QED




$$a_\mu^{(2)} = \frac{\alpha}{2\pi}$$

**Fig. 1** *Lowest Order QED Contribution*

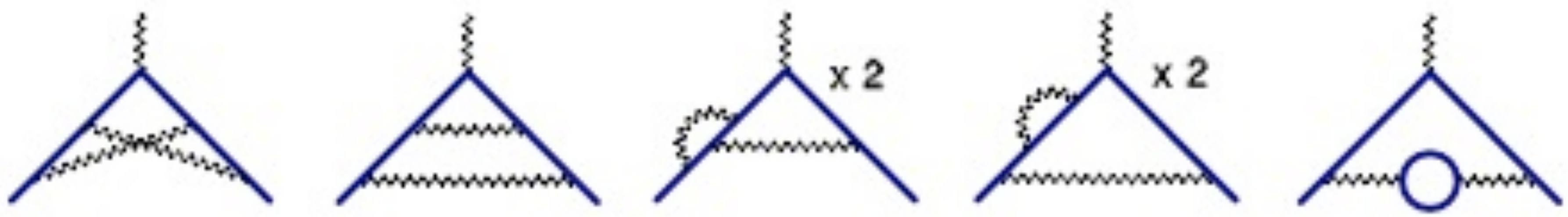
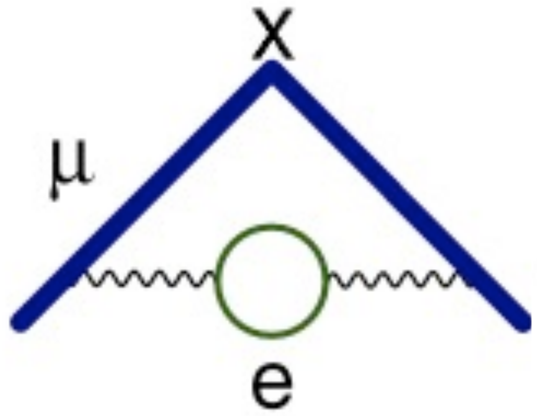



**Fig. 2** *QED Contribution at the Two Loop Level*

$$a_l^{(4)} = \left\{ \frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2} \ln 2 + \frac{3}{4}\zeta(3) \right\} \left(\frac{\alpha}{\pi}\right)^2$$




$$a_\mu = \left[ \underbrace{\left(\frac{2}{3}\right)}_{\beta_1} \left(\frac{1}{2}\right) \log \frac{m_\mu}{m_e} - \frac{25}{36} + \mathcal{O}\left(\frac{m_e}{m_\mu}\right) \right] \left(\frac{\alpha}{\pi}\right)^2$$

**Fig. 3** *Vacuum Polarization contribution from a Small Internal Mass*

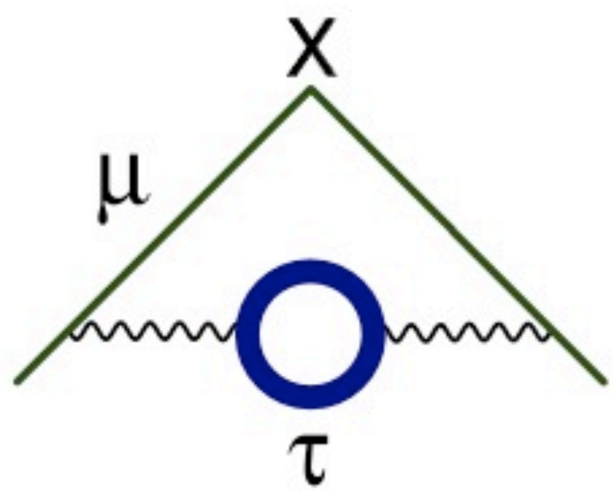



**Fig. 4** *Vacuum Polarization contribution from a Large Internal Mass*

which gives a contribution

$$\alpha(m_\mu) = \left[ \frac{1}{45} \left(\frac{m_\mu}{m_\tau}\right)^2 + \mathcal{O}\left(\frac{m_\mu^4}{m_\tau^4} \log \frac{m_\tau}{m_\mu}\right) \right] \left(\frac{\alpha}{\pi}\right)^2 .$$

# Pure QCD: total

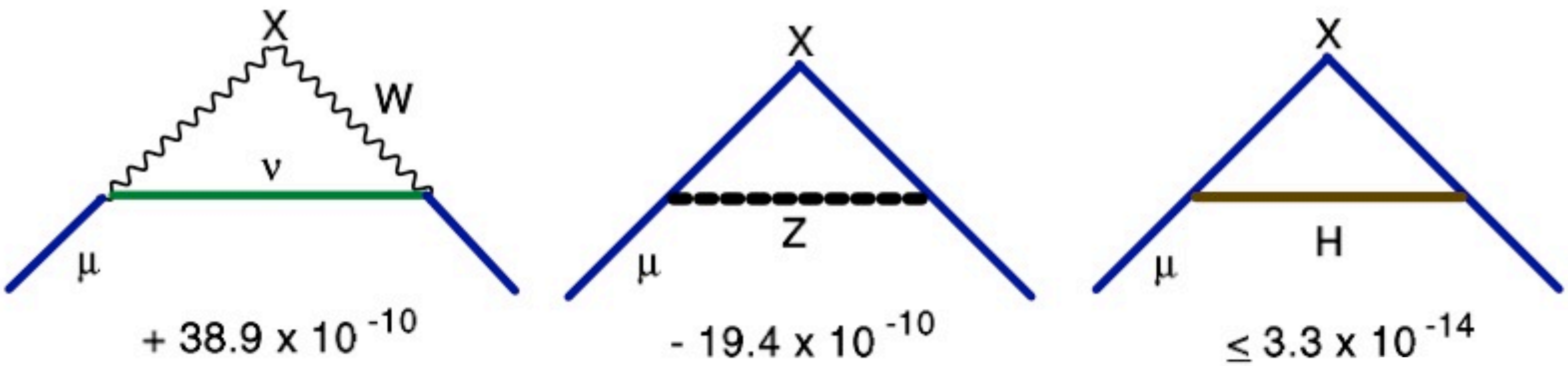
At three loops there are 72–Feynman diagrams of this type which contribute. Quite remarkably, they are also known analytically [8]. They bring in transcendental numbers like $\zeta(3)$, the Riemann zeta–function of argument 3, and of higher complexity.

At the four loop level, there are 891 Feynman diagrams of this type, and their numerical evaluation is still in progress [9].

Altogether, the purely QED contribution to the muon anomalous magnetic moment of the muon, including $e$, $\mu$, and $\tau$ lepton loops is known to an accuracy which is certainly good enough for the present comparison between theory and experiment

$$a_\mu(\text{QED}) = (11\ 658\ 470.57 \pm 0.29) \times 10^{-10}\,. \tag{11}$$

# Electroweak corrections




$19.48 \times 10^{-10}$

**Fig. 12** *Weak Interactions at the one loop level*

1-loop calculations of the Electroweak effect have been done in the early '70s by several groups

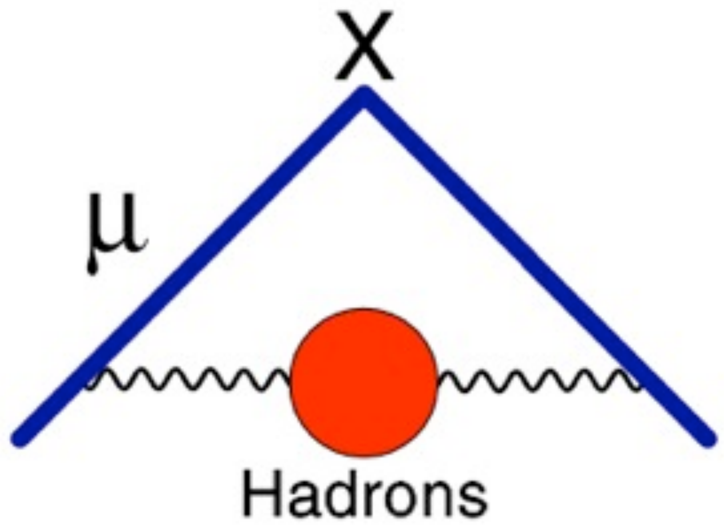
W.A. Bardeen, R. Gastmans and B.E. Lautrup, Nucl. Phys. **B46** (1972) 315; G. Altarelli, N. Cabbibo and L. Maiani, Phys. Lett. **40B** (1972) 415; R. Jackiw and S. Weinberg, Phys. Rev. **D5** (1972) 2473; I. Bars and M. Yoshimura, Phys. Rev. **D6** (1972) 374; M. Fujikawa, B.W. Lee and A.I. Sanda, Phys. Rev. **D6** (1972) 2923.

Result of 1+2 loops

$$a_\mu^{EW} = \frac{G_F}{\sqrt{2}} \frac{m_\mu^2}{8\pi^2} \left[ \frac{5}{3} + \frac{1}{3} \left(1 - 4\sin^2\theta_W\right)^2 - \left(\frac{\alpha}{\pi}\right)(159 \pm 4) \right] = (15.2 \pm 0.1) \times 10^{-10}$$

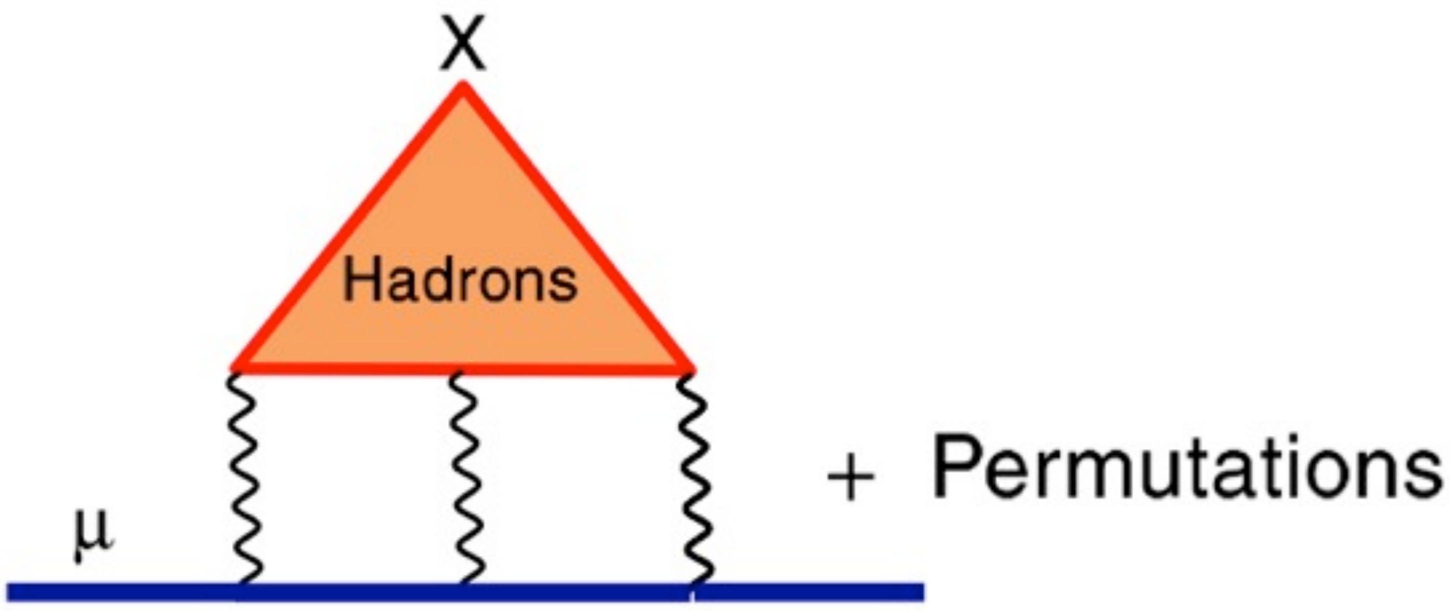
$19.48 \times 10^{-10}$

# Hadronic corrections: vacuum polarization




*Hadronic Vacuum Polarization Contribution*

# Hadronic light-by-light scattering



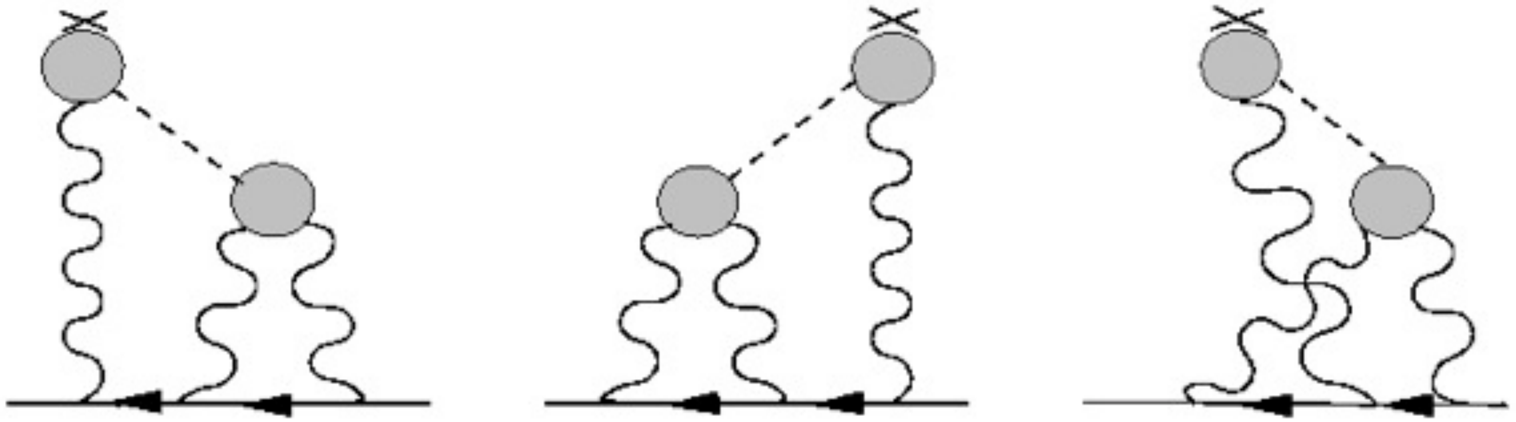



Figure 2: The pion-pole contributions to light-by-light scattering. The shaded blobs represent

# BNL, Muon g-2 Collaboration, 2004
# Positive and negative muons combined

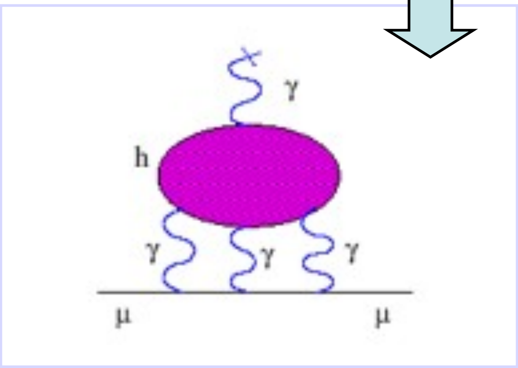
$$\boxed{a_\mu^{\rm exp} \; = \; 11\ 659\ 208(6) \times 10^{-10} \ \ (0.5 \ \ {\rm ppm}).}$$

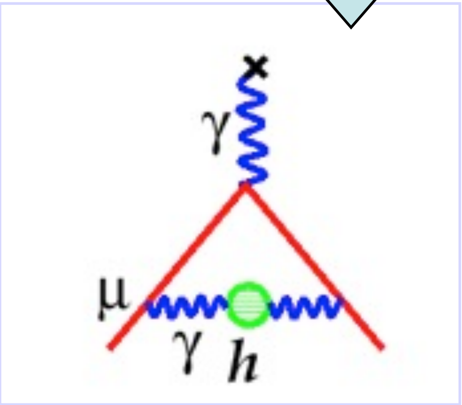
$$\delta^{\rm QED} a_\mu = 11\ 658\ 470.35 \pm 0.28 \times 10^{-10}$$

Electroweak: $(15.4 \pm 0.2) \times 10^{-10}$

Hadron (CMD2 +KLOE): $\delta^{\rm VP}_{\rm had} a_\mu = [(692.4 \pm 6.4)_{\rm lo} - (9.79 \pm 0.095)_{\rm nlo}] \times 10^{-10}$

Light-by-light: $\delta^{\rm lbl}_{\rm had} a_\mu = (8 \pm 4) \times 10^{-10}$




**Total SM:**

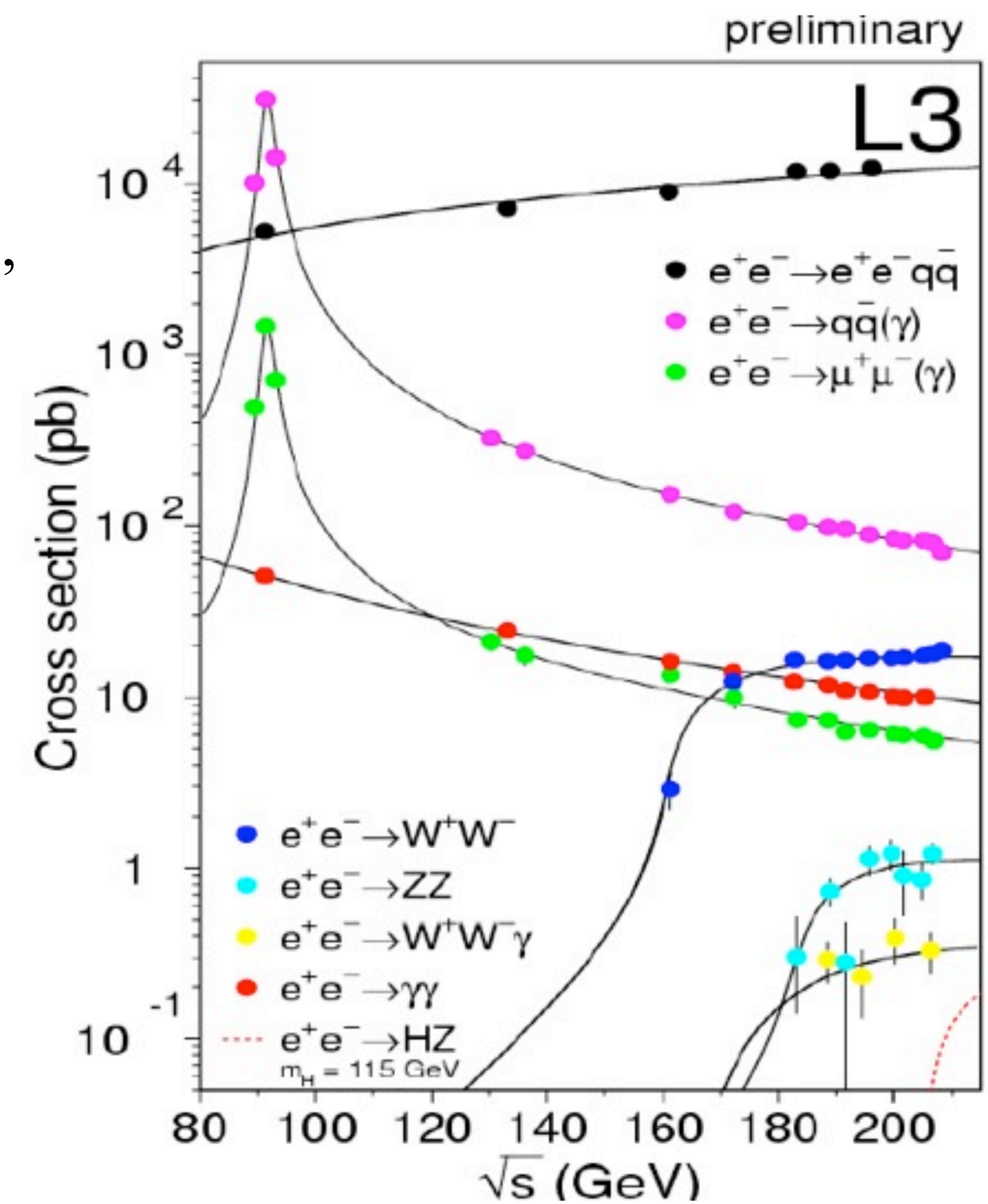
$$\boxed{a_\mu = (11659176.3 \pm 7.4) \times 10^{-10}}$$

$\Delta = \ (32 \pm \ 10) \times 10^{-10}$

K. Hagiwara, A.D. Martin, D. Nomura, and T. Teubner, arXiv:hep-ph/0312250.

# Conclusions: Spinor Electro Dynamics

- Originally a theory of electrons and photons;
- μ and τ-particles behave the same;
- e, μ and τ numbers separately conserved;
- even in world made by electrons and photons only, $e^+ e^-$ annihilation gives access to muons, τ and to all other charged fermions.
- Spinor QED is determined by few general principles:
  - Lorentz invariance
  - Gauge invariance
  - matter particles are fermions with spin 1/2 (Dirac particles)
  - renormalizability
- The prototype of a fundamental field theory and an extraordinary success
- At high energy, it merges into the Unified Electroweak Theory, which is as renramlizable as QED, thanks to the Brout-Englert-Higgs mechanism, which is now under control.



# A corollary: dwarfs of the shoulder of giants


- the construction of QED is the work of giants
- if we want to see further than they, we need to stay firmly on their shoulders
- before launching a new theory of the world, ***I beg you to check*** that your theory is compatible with the QED calculation of the electron anomaly

$$a_e(\text{exp}) = 1\ 159\ 652\ 180.73\ (0.28) \times 10^{-12} \quad [0.24\ ppb]$$

$$a_e(\text{Rb}) = 1\ 159\ 652\ 182.79\ (0.11)(0.37)(7.72) \times 10^{-12},$$

$$a_e(\text{exp}) - a_e(\text{Rb}) = -2.06\ (7.72) \times 10^{-12}.$$

If not, abstain!