7) Additività rispetto al dominio di integrazione.

Se $D$ dominio normale è unione di due domini normali $D_1$ e $D_2$ senza punti interni in comune, e se $f \in \mathcal{R}(D)$, allora

$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} f(x,y)\,dx\,dy + \iint_{D_2} f(x,y)\,dx\,dy.$$

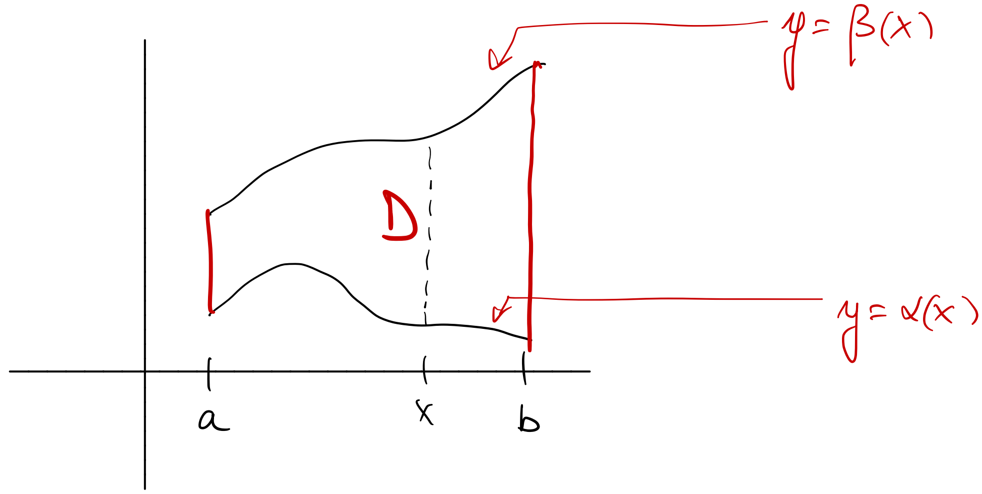
OGGI ESERCITAZIONE FACOLTATIVA ore 16:00
AULA III DIP. MATEMATICA

# Formule di riduzione per il calcolo di integrali doppi.

D dominio x-normale

$$D = \{ (x,y) \in \mathbb{R}^2 : \ a \le x \le b, \ \alpha(x) \le y \le \beta(x) \}$$

$\alpha, \beta : [a,b] \to \mathbb{R}$ continue e t.c. $\alpha(x) \le \beta(x)$ $\forall x$




$f(x,y) : D \to \mathbb{R}$ funzione continua in D. Allora vale la seguente formula.

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \Big( \underbrace{\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy}_{\text{Questa è una funzione continua della } x} \Big) dx$$

Ovviamente se D è y-normale, cioè

$$D = \{ (x,y) : \ c \le y \le d, \ \gamma(y) \le x \le \delta(y) \}$$

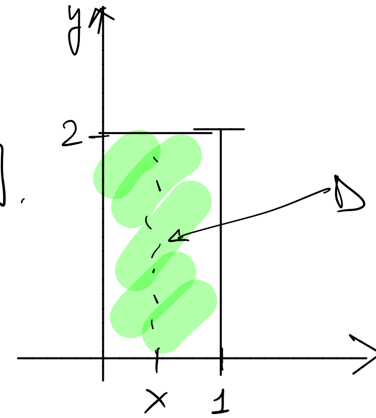
$$\iint_D f(x,y)\,dx\,dy = \int_c^d \Big( \int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx \Big) dy$$

Se il dominio è normale rispetto a entrambi gli assi, valgono entrambe le formule, che devono dare lo stesso risultato.

**ESEMPIO** Calcolare

$$\iint_D \frac{x}{1+y}\,dx\,dy \qquad \text{dove } D = [0,1]\times[0,2].$$

continua in $D$.

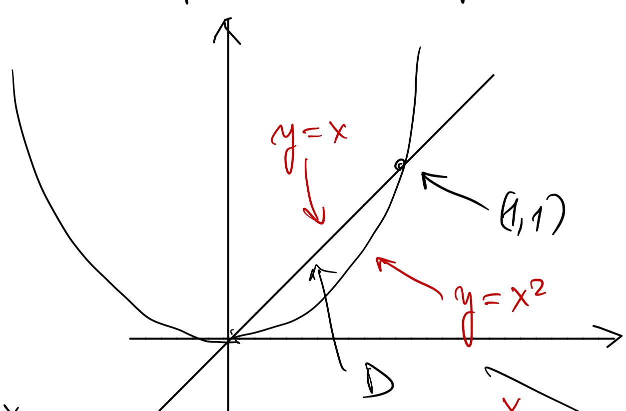



$$\iint_D \frac{x}{1+y}\,dx\,dy = \int_0^1 dx \left( \int_0^2 dy \, \frac{x}{1+y} \right) =$$

$$= \int_0^1 dx \; x \, \ln|1+y| \Big|_{y=0}^{y=2} = \int_0^1 dx \; x \ln 3 = \frac{\ln 3}{2}$$

**ESEMPIO** Calcolare $\displaystyle\iint_D (2x+3y)\,dx\,dy$

dove $D$ è la regione di piano compresa tra la parabola $y = x^2$ e la retta $y = x$




OSS $D$ è un dominio $x$ normale

$$D = \{(x,y) : 0 \le x \le 1, \; x^2 \le y \le x\}$$

$$\iint_D (2x+3y)\,dx\,dy = \int_0^1 dx \left( \int_{x^2}^{x} dy \, (2x+3y) \right) = \qquad \cancel{\int_{x^2}^{x} dy \left( \int_0^1 dx \, (2x+3y) \right)}$$

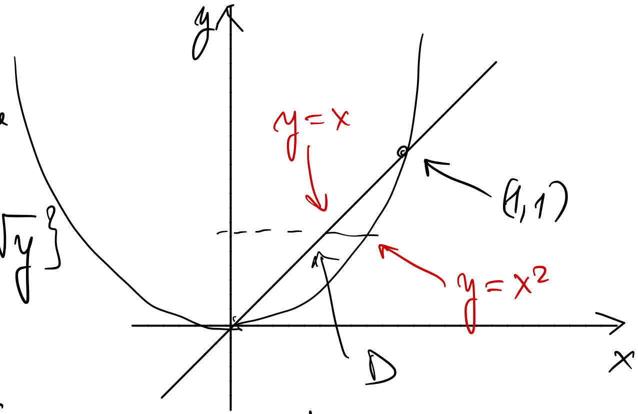
$$= \int_0^1 dx \left[ 2x(x-x^2) + \frac{3}{2} y^2 \Big|_{y=x^2}^{y=x} \right] = \int_0^1 \left( 2x^2 - 2x^3 + \frac{3}{2}x^2 - \frac{3}{2}x^4 \right) dx =$$

$$= \left( \frac{7}{6}x^3 - \frac{x^4}{2} - \frac{3}{10}x^5 \right) \Big|_0^1 = \frac{7}{6} - \frac{1}{2} - \frac{3}{10} = \frac{35-15-9}{30} = \frac{11}{30}$$

OSS. importante:

D è anche un dominio y-normale

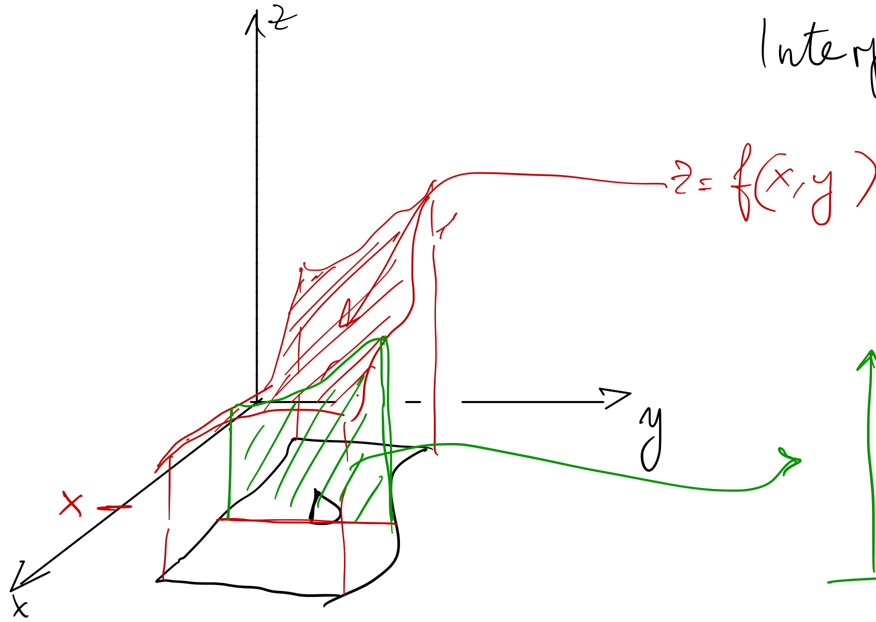
$$D = \{(x,y):\ 0 \le y \le 1,\ y \le x \le \sqrt{y}\}$$

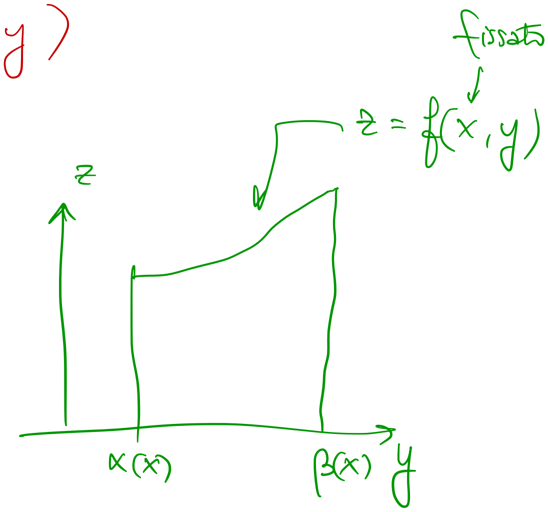



$$\iint_D (2x+3y)\,dx\,dy = \int_0^1 dy \int_y^{\sqrt{y}} dx\,(2x+3y) = \int_0^1 dy \left( x^2\Big|_{x=y}^{x=\sqrt{y}} + 3y(\sqrt{y}-y)\right) =$$

$$= \int_0^1 dy \left(y - y^2 + 3y^{3/2} - 3y^2\right) = \frac{1}{2} - \frac{4}{3} + 3\cdot\frac{2}{5} = \frac{15-40+36}{30} = \frac{11}{30}$$

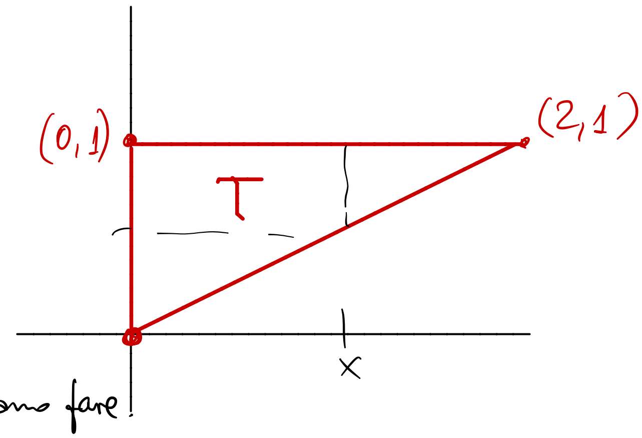
# Interpretazione grafica




## ESERCIZIO

**ESERCIZIO.** Calcolare $\iint_T e^{y^2}\,dx\,dy$ dove $T$ è il triangolo di vertici $(0,0)$, $(0,1)$ e $(2,1)$




$$T=\left\{(x,y):\ 0\le x\le 2,\ \frac{x}{2}\le y\le 1\right\}$$

$$\iint_T e^{y^2}\,dx\,dy=\int_0^2 dx\int_{x/2}^1 dy\ e^{y^2}=$$ non lo sappiamo fare!

Proviamo a invertire l'ordine di integrazione.

$$T=\{(x,y):\ 0\le y\le 1,\quad 0\le x\le 2y\}$$

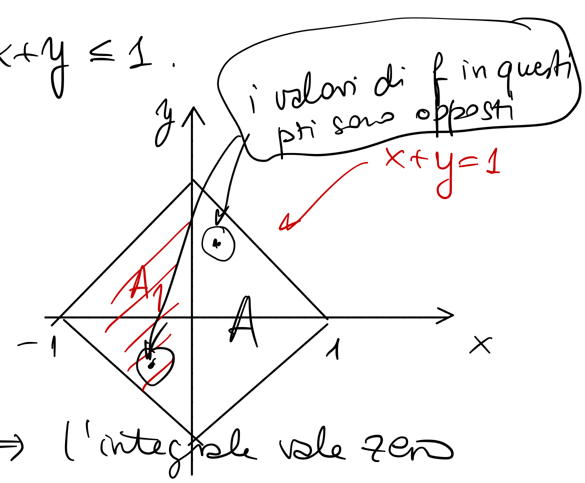
$$\iint_T e^{y^2}\,dx\,dy=\int_0^1 dy\left(\int_0^{2y}dx\ e^{y^2}\right)=\int_0^1 dy\ e^{y^2}\,2y=e^{y^2}\Big|_0^1=e-1$$

<u>OSS</u> Quindi scegliere bene l'ordine di integrazione è (a volte) cruciale!

**ESERCIZIO** $\iint_A (x+y)\,e^{x^2+y^2}\,dx\,dy$ $(=0)$

$A=\{(x,y):\ |x|+|y|\le 1\}$. dominio simmetrico rispetto all'asse $x$ e all'asse $y$.

Se prendo $x\ge 0$, $y\ge 0$, diventa $x+y\le 1$.



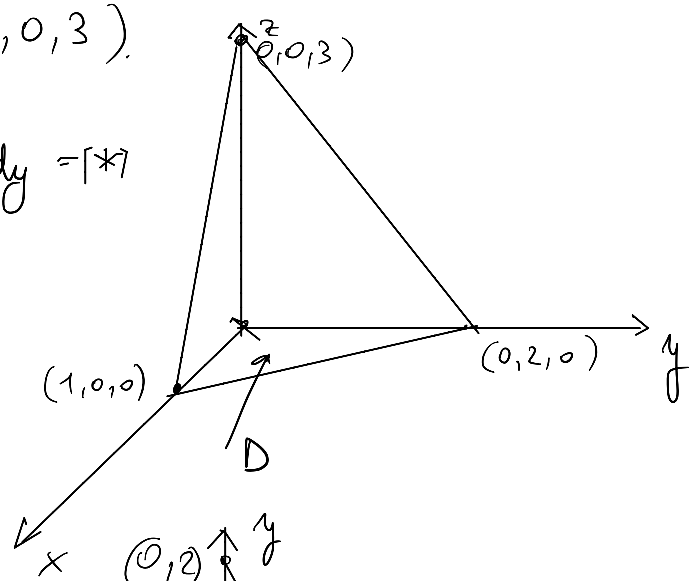

$A=A_1\cup A_2$

$A_1=\{(x,y):-1\le x\le 1,\ -1-x\le y\le 1+x\}$

$A_2=\{(x,y):\ 0\le x\le 1,\ -1+x\le y\le 1-x\}$

Ma non serve integrare su $A_1$ e $A_2$, perché $f(-x,-y)=-f(x,y)$ $\Rightarrow$ l'integrale vale zero

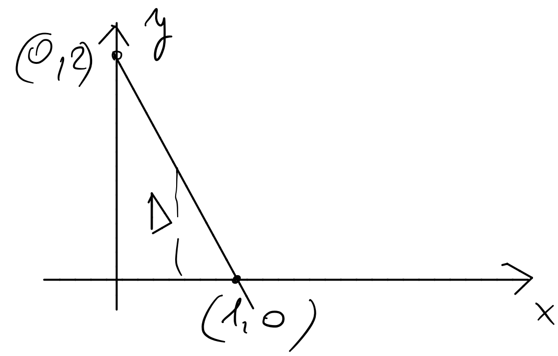
**ESERCIZIO** Calcolare il volume del tetraedro di vertici $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, $(0,0,3)$.




$$\mathrm{Vol}(\text{tetraedro}) = \iint_D \left(3 - 3x - \frac{3}{2}y\right) dx\,dy = [*]$$

$$z = 3 + \overset{-3}{\cancel{a}}\,x + \overset{-3/2}{\cancel{b}}\,y$$

dove $D$ è il triangolo del piano $xy$ di vertici $(0,0)$, $(1,0)$ e $(0,2)$

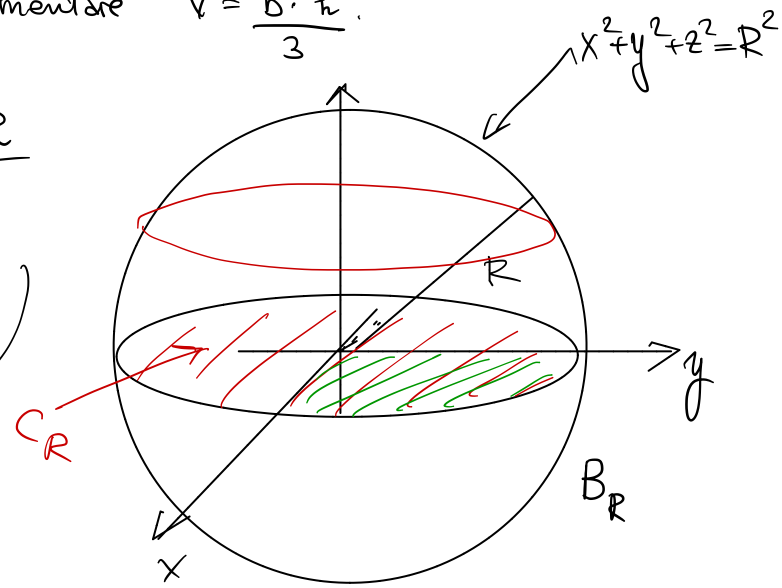



$$[*] = 3\int_0^1 dx \int_0^{2-2x} dy \left(1 - x - \frac{y}{2}\right) =$$

$$= 3\int_0^1 dx \left[2(1-x)^2 - \frac{1}{4}(2-2x)^2\right] = 3\int_0^1 dx\,(1-x)^2 = \frac{3}{3}(1-x)^3\Big|_1^0 = 1$$

in accordo con la formula elementare $V = \frac{b \cdot h}{3}$.

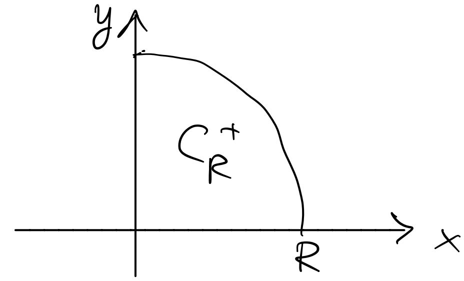
## Volume di una palla di raggio $R$




$$\mathrm{Vol}(B_R) = 2\iint_{C_R} \sqrt{R^2 - x^2 - y^2}\,dx\,dy =$$

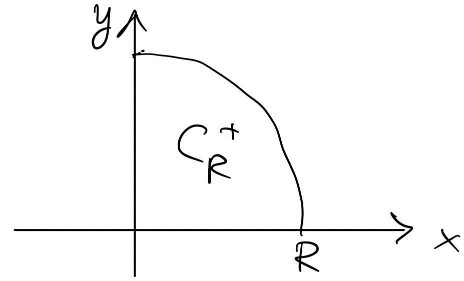
$$C_R = \{(x,y) : x^2 + y^2 \le R^2\}$$

$$= (\text{per simmetria}) = 8\iint_{C_R^+} \sqrt{R^2 - x^2 - y^2}\,dx\,dy$$




$$C_R^+ = \{(x,y) : 0 \le x \le R,\ 0 \le y \le \sqrt{R^2 - x^2}\}$$

$$\mathrm{Vol}\,(B_R) = 8 \iint_{C_R^+} \sqrt{R^2-x^2-y^2}\,dx\,dy = (*)$$




$$C_R^+ = \{(x,y) : 0 \le x \le R,\ \ 0 \le y \le \sqrt{R^2-x^2}\}$$

$$(*) = 8 \int_0^R dx \int_0^{\sqrt{R^2-x^2}} dy \ \sqrt{R^2-x^2-y^2} = (*)$$

$$\int dy\, 1 \sqrt{a^2-y^2} = y\sqrt{a^2-y^2} + \int \frac{y^2-a^2+a^2}{\sqrt{a^2-y^2}}\,dy =$$

$$= y\sqrt{a^2-y^2} - \int \sqrt{a^2-y^2}\,dy + \int \frac{a^2}{\sqrt{a^2-y^2}}\,dy$$

$$// \int \frac{a}{\sqrt{1-\left(\frac{y}{a}\right)^2}}\,dy = a^2 \operatorname{arcsen}\left(\frac{y}{a}\right)$$

$$\int dy \sqrt{a^2-y^2} = \frac{1}{2}\, y\sqrt{a^2-y^2} + \frac{a^2}{2} \operatorname{arcsen}\left(\frac{y}{a}\right)$$

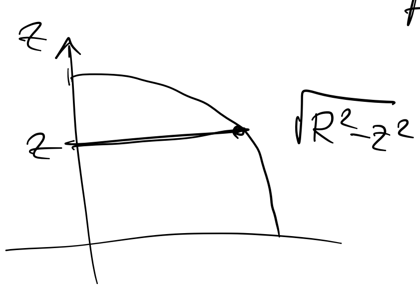
$$a = \sqrt{R^2-x^2}$$

$$(*) = 4 \int_0^R dx \left( \cancel{y\sqrt{R^2-x^2-y^2}} + (R^2-x^2) \operatorname{arcsen} \frac{y}{\sqrt{R^2-x^2}} \right) \Bigg|_{y=0}^{y=\sqrt{R^2-x^2}}$$

$$= 4 \int_0^R \frac{\pi}{2}(R^2-x^2)\,dx = 2\pi \left( R^3 - \frac{R^3}{3} \right) = 2\pi \cdot \frac{2}{3} R^3 = \frac{4}{3}\pi R^3$$

Altro metodo: usando il principio di Cavalieri. Fissato $z \in (0,R)$ considero l'insieme $A(z) = \{(x,y) : (x,y,z) \in B_R\} =$

= cerchio di centro $(0,0)$ e raggio $\sqrt{R^2-z^2}$



$$\mathrm{Vol}\, B_R = 2\int_0^R dz\, (\text{area } A(z)) = 2\int_0^R dz\ \pi (R^2 - z^2) =$$

$$= 2\pi \left(R^3 - \frac{R^3}{3}\right) = \frac{4}{3}\pi R^3.$$

---

## BARICENTRO DI DOMINI PIANI

$D$ dominio normale

$\mu(x,y)$ = densità superficiale della lastra.

Baricentro di $D$ = $B(x_B, y_B)$

$$x_B = \frac{1}{m(D)} \iint_D x\, \mu(x,y)\, dx\, dy = \frac{\iint_D x\, dx\, dy}{\text{area } D}$$

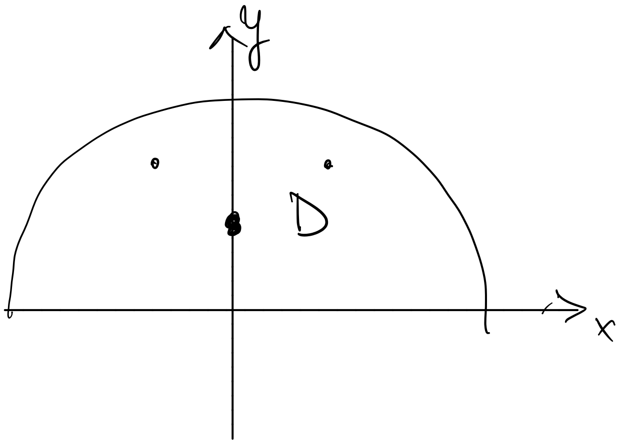
se $\mu$ è costante

$$m(D) = \iint_D \mu(x,y)\, dx\, dy$$

$$y_B = \ldots \iint_D y\, \mu(x,y)\, dx\, dy = \frac{\iint_D y\, dx\, dy}{\text{area } D}$$

## BARICENTRO DI UN SEMICERCHIO (Sottinteso: $\mu$ = cost).

# BARICENTRO DI UN SEMICERCHIO (Sottinteso: $\mu = \text{cost}$).




$$D = \{(x,y) : y \geq 0, \ x^2 + y^2 \leq R^2\}$$

$$x_B = \frac{2}{\pi R^2} \iint_D x \, dx\,dy = 0$$

$$y_B = \frac{2}{\pi R^2} \iint_D y \, dx\,dy = \frac{\cancel{2}\,4}{\pi R^2} \int_0^R dx \int_0^{\sqrt{R^2-x^2}} dy \ y = \frac{2}{\pi R^2} \int_0^R dx \left(R^2 - x^2\right) =$$

$$= \frac{2}{\pi R^2} \, \frac{2}{3} R^3 = \frac{4}{3\pi} R$$