

Measurements for industrial sustainability and environmental safety

PART I Lesson 4

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- **RAPIDITY (Dynamic Response)**

So far the ***measurand*** (the *physical quantity* for which we wish to *measure the intensity*) has been considered strictly CONSTANT during the whole measurement procedure !

From here on it will change its intensity with time during the measurement ... the *measurand* becomes a *function of time* !



both the ***input measurand*** and the ***output measurement*** are functions of time $\rightarrow i(t) \ u(t)$

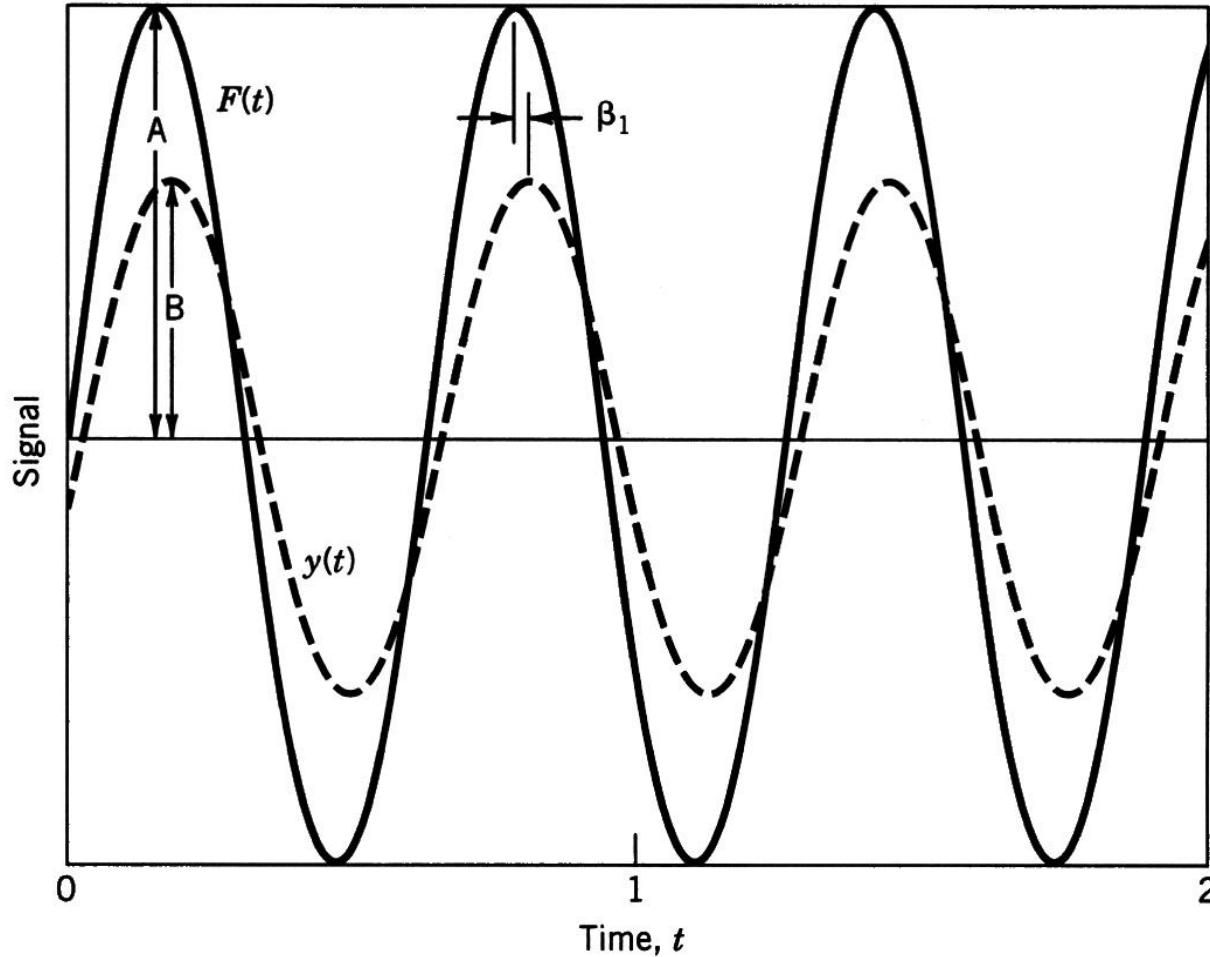
We may call **rapidity of an instrument** “the attitude to *correctly follow the changes of the measurand* during time”

Rapidity of *mechanical instruments* is always limited by the **inertia** and the **damping effects** of its moving parts !

Rapidity for *electronic instruments* is always limited by the combination of its **capacitive** and **inductive reactances** !

An instrument with **insufficient rapidity** (***insufficient dynamic response***) during a measurement will output a *measurement waveform* which will be attenuated and out of phase (delayed) with respect to the measurand !

The *output measurement waveform* will be ***distorted*** with respect to the *input measurand waveform* ...



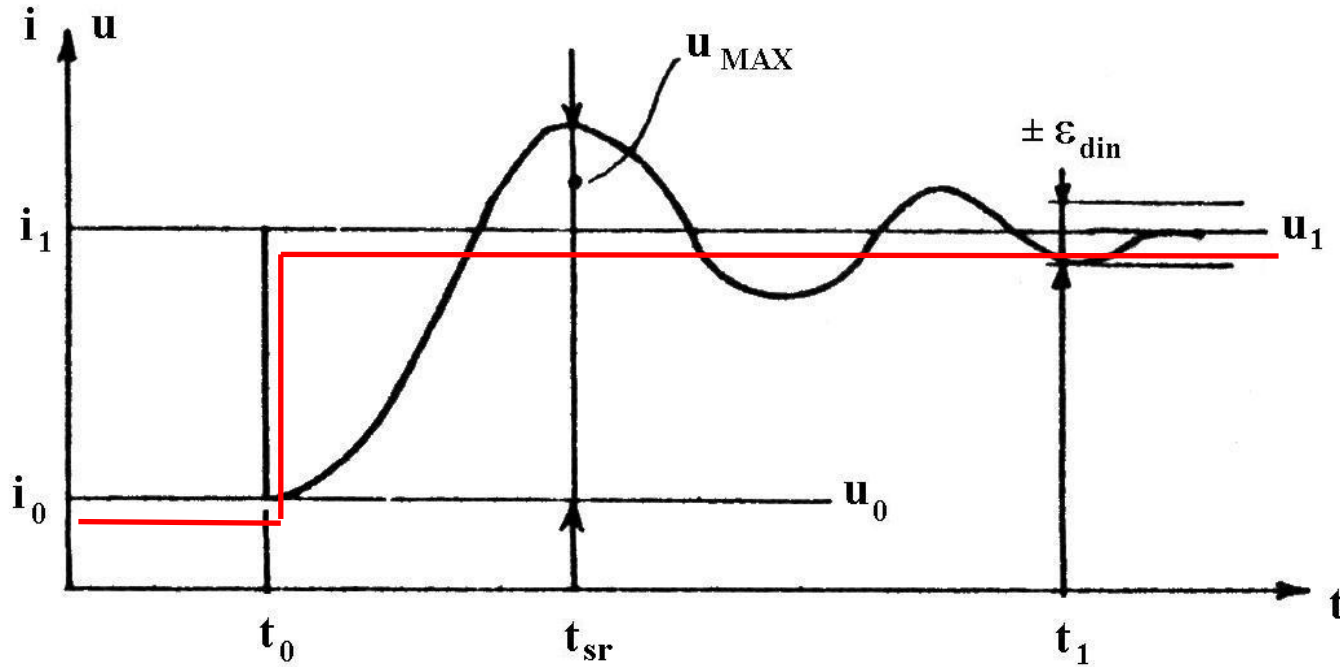
Relationship between a sinusoidal input and output:
amplitude, frequency, and time lag.

We have basically two schemes or methods to study the dynamic response of a measurement instrument:

1. **STEP RESPONSE** with which we can get some important parameters such the settling time and the time constant
2. **FREQUENCY RESPONSE** with which we can get other important parameters such the cut-off frequency and the damping factor ...

There is also a third scheme, which is used less often: the **ramp response** that leads to the delay time ...

Step Response



At time t_0 the input signal changes instantaneously its value from i_0 to i_1

The instrument output will try to follow this change, from u_0 to u_1 , the best he can...

However, it will take some time $t_s = t_1 - t_0$ to reach the correct final value within a certain error $\pm \epsilon_d$!

$\pm \epsilon_d$ is the **dynamic error**, which has to be established in advance by the operator ...

$t_s = t_1 - t_0$ is the **settling time** of the instrument, which provides a first idea of its dynamic response

$t_{SLEW} = t_{sr} - t_0$ is the **slew rate**, which is the time the instrument takes to reach the first overshoot peak...

... not all the instruments show an overshoot during the step response !

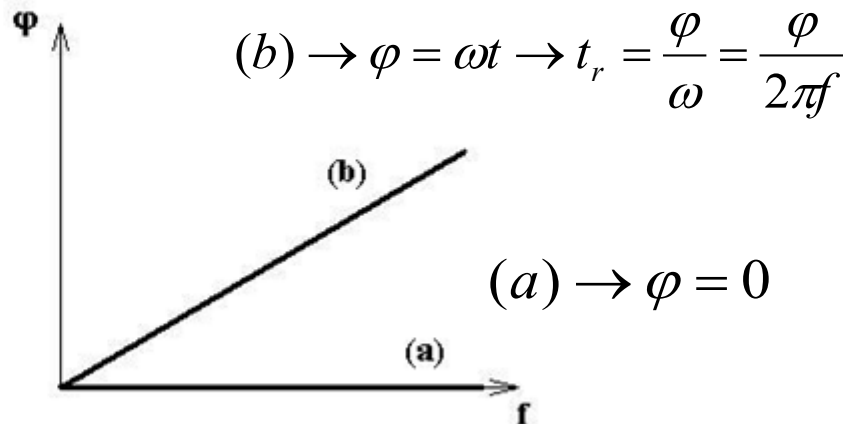
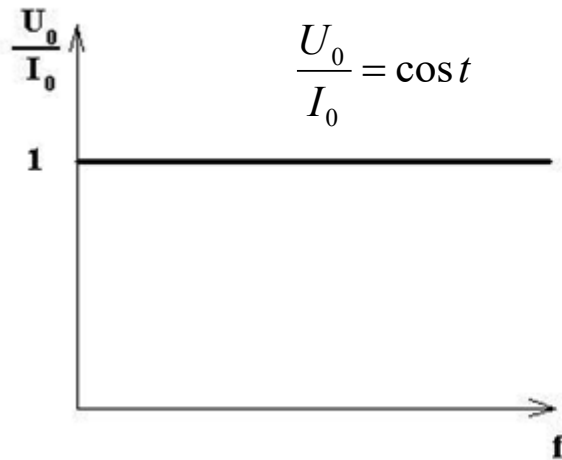
Frequency Response

In the more general case, an input variable (*measurand*) changes “periodically” during time, then we can refer to simple sinusoidal inputs because every periodic signal can be decomposed by the **Fourier Series**:

$$f(x) = f(x + 2\pi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

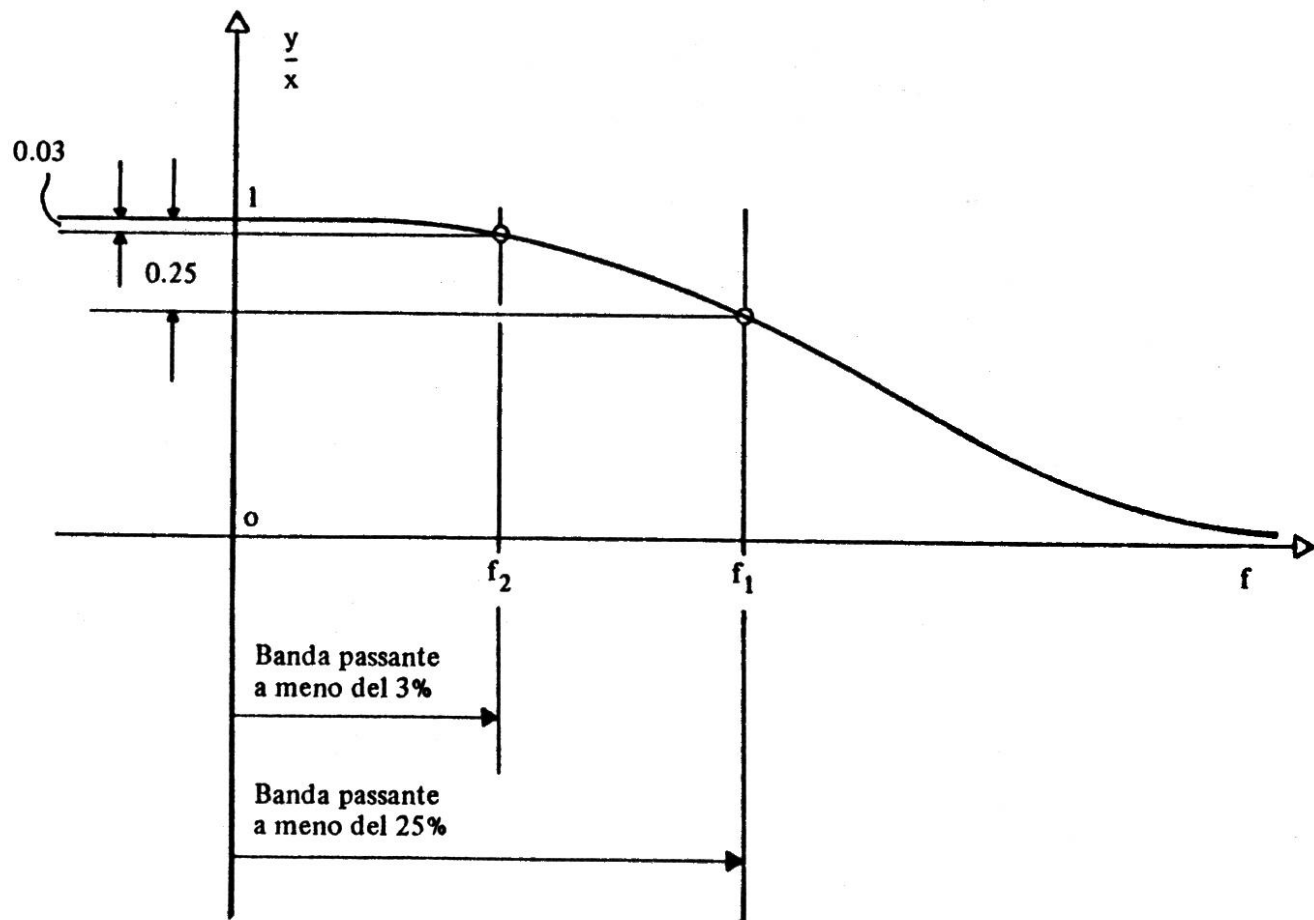
input: $i(t) = I_0 \sin \omega t$ with $\omega = 2\pi f$ **frequency**

output: $u(t) = U_0 \sin(\omega t + \varphi)$ with φ **phase delay**



ideal **frequency response**
and **phase response** curves:

- constant **output/input ratio** for “every frequency” of the input !
- zero **phase delay** for “every frequency” of the input !



Real instrument **frequency response** example:

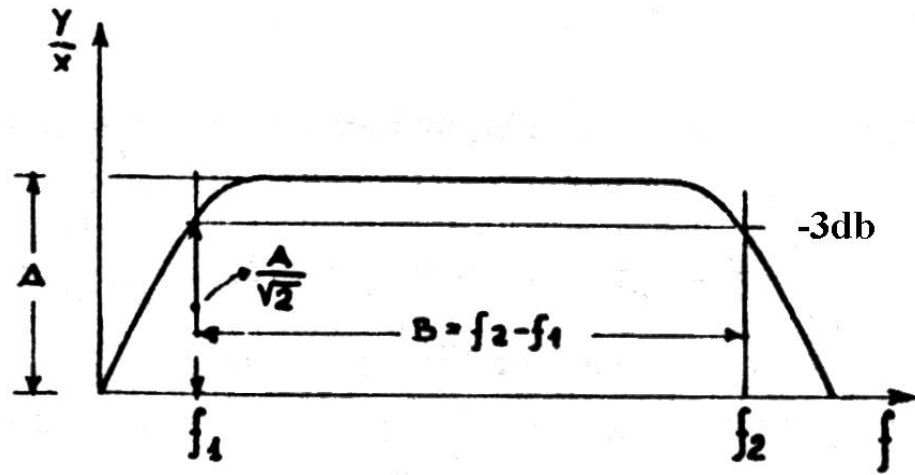
Real instruments always *attenuate the output amplitude* for higher frequencies ($f \rightarrow \infty$) because:

- mechanical instrument have inner moving parts with inertia which can not have “infinite acceleration”
- electronic instrument have inner components which can not have
inductive reactance $X_L = j\omega L = \infty$
capacitive reactance $X_C = 1/j\omega C = 0$

The extension of an *instrument frequency response* depends also from the **dynamic error** the operator accepts:

- if we accept a dynamic error $\varepsilon_d = 3\%$ we have a **cut-off frequency** f_2
- If we accept a dynamic error $\varepsilon_d = 25\%$ we have a much higher **cut-off frequency** $f_1 > f_2$

All frequencies between f_0 and f_1 (or f_2) form the **instrument bandwidth or pass-band** !



There are many instruments where the bandwidth does “not start” from frequency $f_0 = 0$ Hz, but starts at a higher value f_1 . In these cases, we have “two cut-off frequencies” and the pass-band is between these two frequencies: $B = f_{ts} - f_{ti}$

Most of the times the *dynamic error* (which can be “fairly large”) is expressed in a *logarithmic scale*, inherited from “acustics” studies and the logarithmic unit is the **decibel** :

It is world wide accepted that a tolerable dynamic error limit is $-3dB = 20\log_{10} 0.707$ which means the instrument output **A** is “attenuating the input” to the 70,7 % of its real amplitude or that it is doing a **dynamic error** of 29,3 % !!



We change now notation for the *input* and *output variables*:

$y(t) \rightarrow$ input variable (*measurand*)

$x(t) \rightarrow$ output variable (*instrument response or indicator deflection*)

We say an instrument is dynamically linear if we can write its “dynamic characteristic equation” as follows :

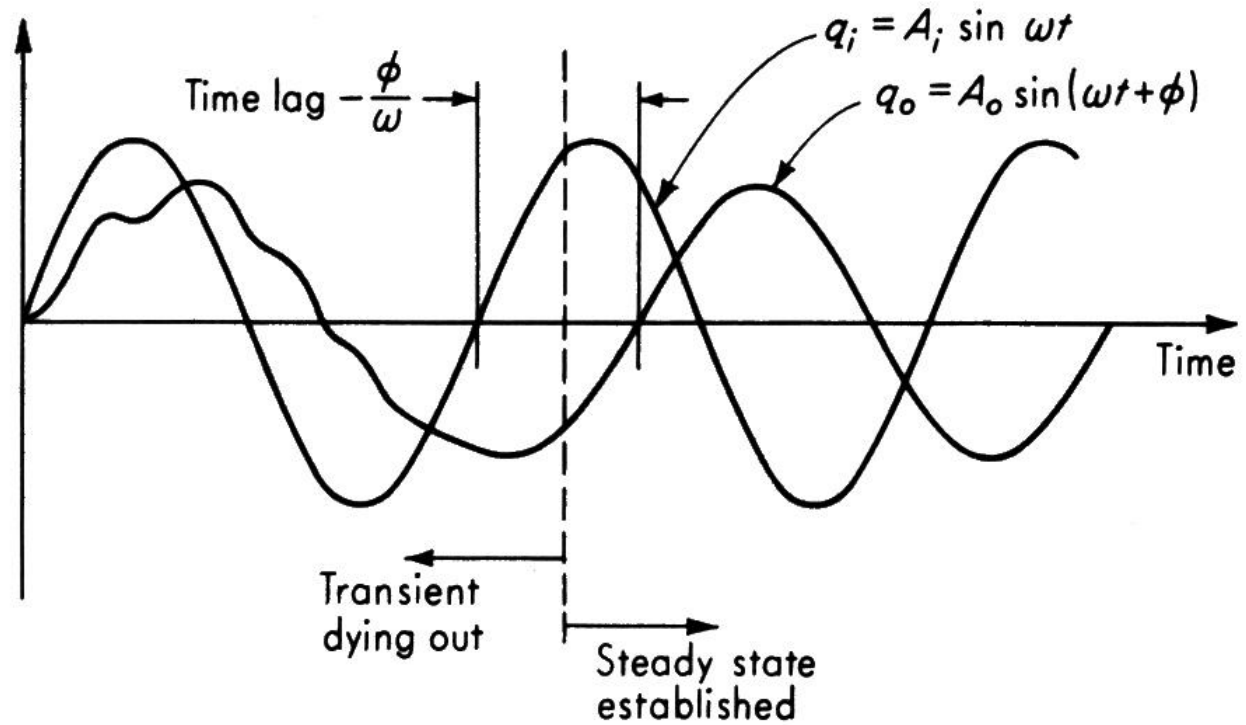
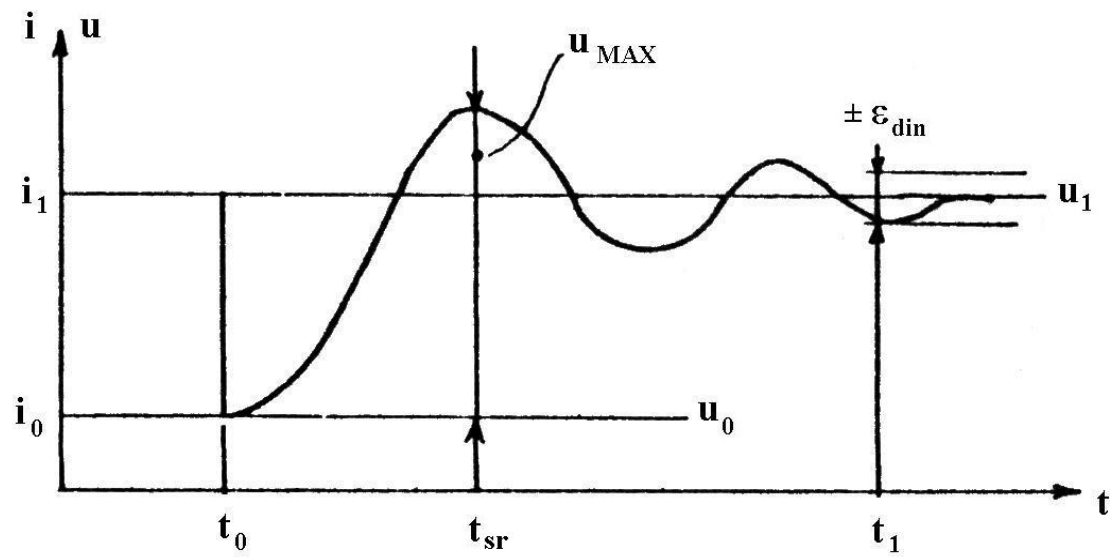
$$a \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + cx = y(t)$$

The instrument dynamic can be described by an ordinary differential equation with *constant coefficients* !
The solution can be written as:

$$x(t) = x_{tr}(t) + x_{rg}(t)$$

Solution of the *associated homogeneous equation* which represents: $x_{tr}(t)$
the transient dynamic response of the instrument

Solution of a *particular integral* of the equation which represents: $x_{rg}(t)$
the dynamic steady state of the instrument



To study the TWO different situations we will employ the TWO different experimental schemes or tools introduced before:

1. For the ***dying out transient*** → the **STEP RESPONSE** $x_{tr}(t)$
2. For the ***established steady state*** → the **FREQUENCY RESPONSE** $x_{rg}(t)$

Zero-order instruments

When the differential equation of the dynamic characteristic has the simplest form, without the derivative terms:

$$a \cdot x = b \cdot y \quad \rightarrow \quad x = \frac{b}{a} y$$

The output signal (or deflection) is “always” proportional to the input signal !
There is no delay between output and input therefore, zero-order instruments “respond instantaneously” and can be considered *ideal instrument* !

Example:

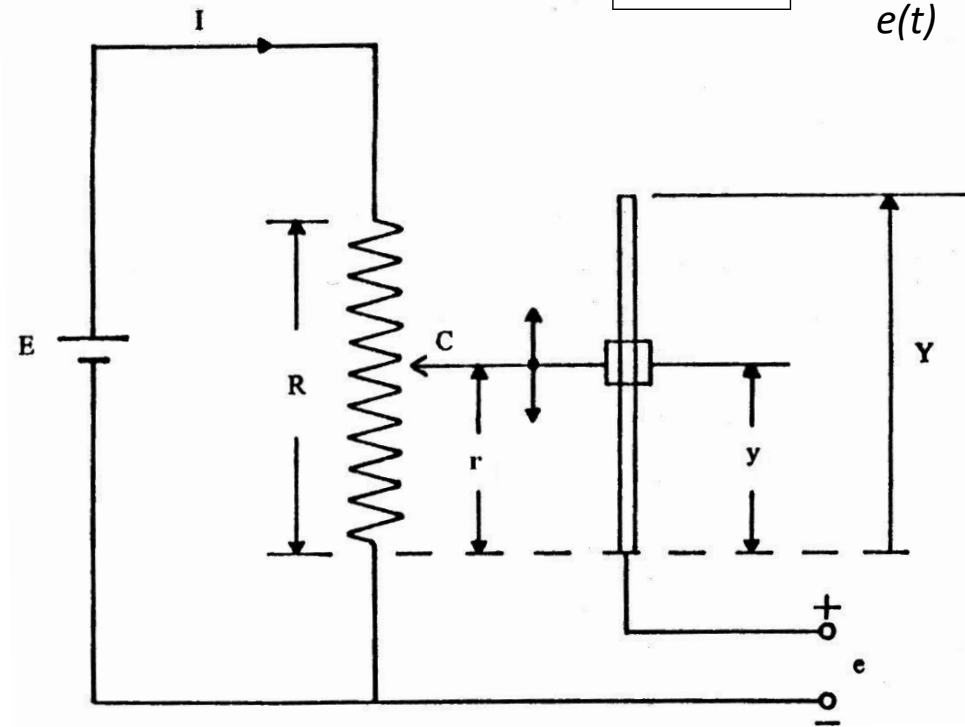


For the **potentiometer** (as a position transducer) we have for the left circuit $E = RI$, while the displacement y is indicated by the output voltage $e(t)$ which is:

$$e(t) = r(t)I = r(t)\frac{E}{R} = \frac{r(t)}{R} E = \frac{\rho \cdot y(t) / S}{\rho \cdot Y / S} E = \frac{y(t)}{Y} E$$

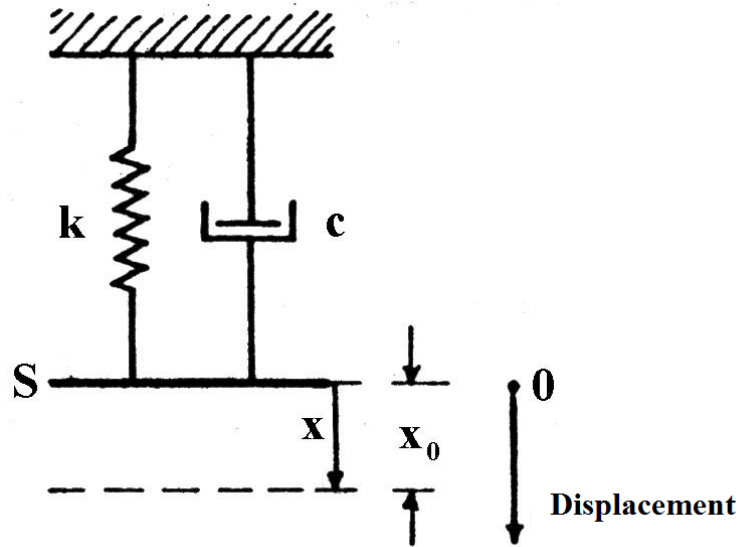
where $R = \rho \frac{Y}{S}$ and $r(t) = \rho \frac{y(t)}{S}$ the variable “time” is only present as “independent variable” for the position $y = y(t)$

Note that $e = \frac{E}{Y} y$ is also the graduation curve (**static characteristic curve**) of the potentiometer and $\frac{de}{dy} = \frac{E}{Y}$ is the **static sensitivity** !



First-order instruments

A **first-order instrument** can “store energy” in only one of its inner elements !



If we displace the indicator S of the simple mechanical system to a position $x_0 \neq 0$, *elastic* (potential) *energy* is stored in the spring k .

If at $t = 0$ we leave the indicator S free of moving, the elastic energy stored in the spring will pull back the indicator to the rest position $x = 0$. During movement, the damper c will oppose the motion.

$$c \frac{dx}{dt} + kx = 0$$

For every time instant t the equation of *force equilibrium* is: $kx = -c\dot{x}$ or $c\dot{x} + kx = 0$

For every time instant t the *velocity* is: $\dot{x} = -\frac{k}{c}x$ therefore the coefficient $\left[\frac{c}{k}\right] = [t]$ is a time !

$$\frac{c}{k} = \lambda$$

is the **time constant** of the first-order instrument of above !

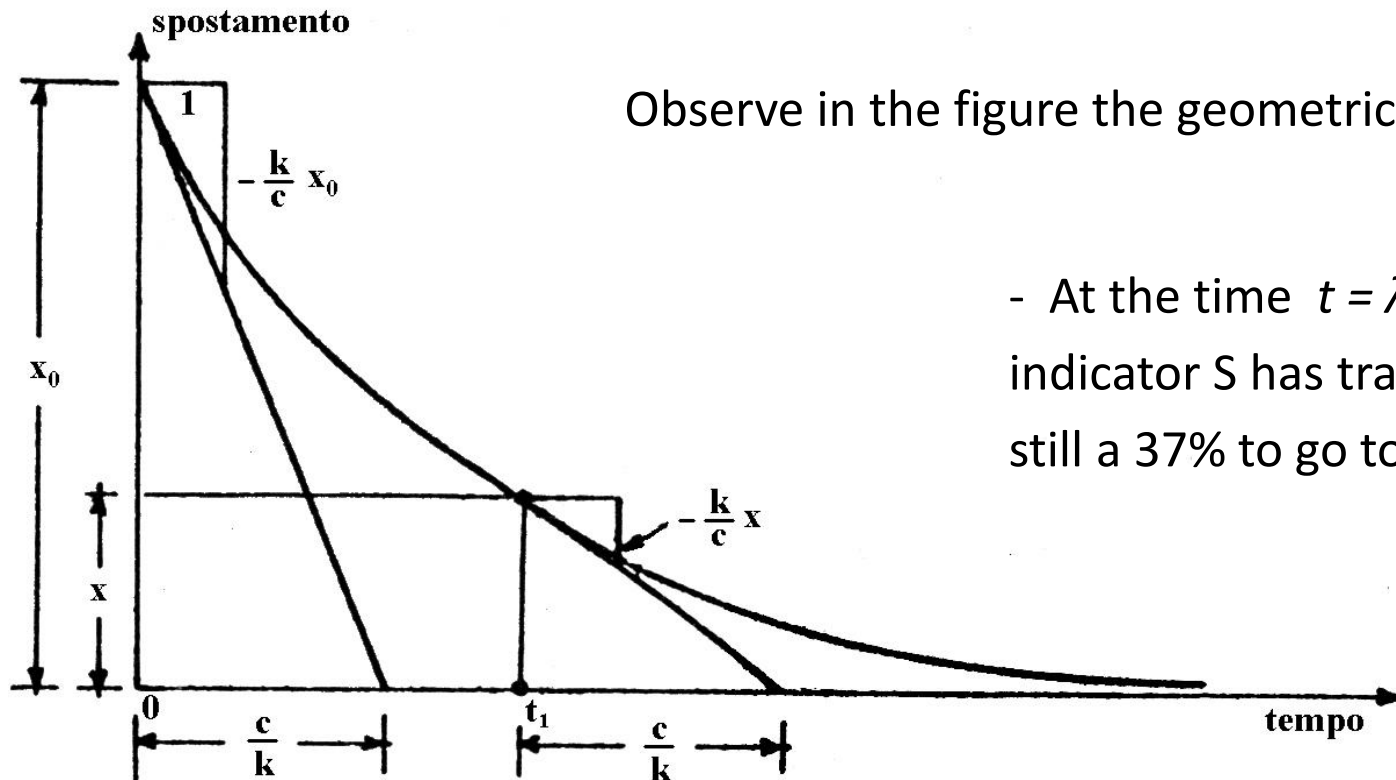
The solution of the 1st order differential equation $c\dot{x} + kx = 0$ is the well known “decreasing exponential”:

$x(t) = x_0 \cdot e^{-\frac{k}{c}t}$ the curve of which can be drawn with the *initial condition*: $t = 0 \rightarrow x(0) = x_0$

- The tangent line to the curve in the point $t = 0$ is: $\dot{x}(0) = -\frac{k}{c}x_0 = -\frac{x_0}{c/k}$

Observe in the figure the geometrical meaning of the time constant $\lambda = \frac{c}{k}$

- At the time $t = \lambda$ we have that $x(\lambda) = x_0 e^{-1} = 0.37 \cdot x_0$ the indicator S has travelled for 63% of the displacement and has still a 37% to go to reach its final position $x = 0$!!



... but these are only introductory definitions and considerations ...

In the general case of the **step response**, an *input variable* is present : a step force F_0 is applied to the indicator S at time $t = 0$...

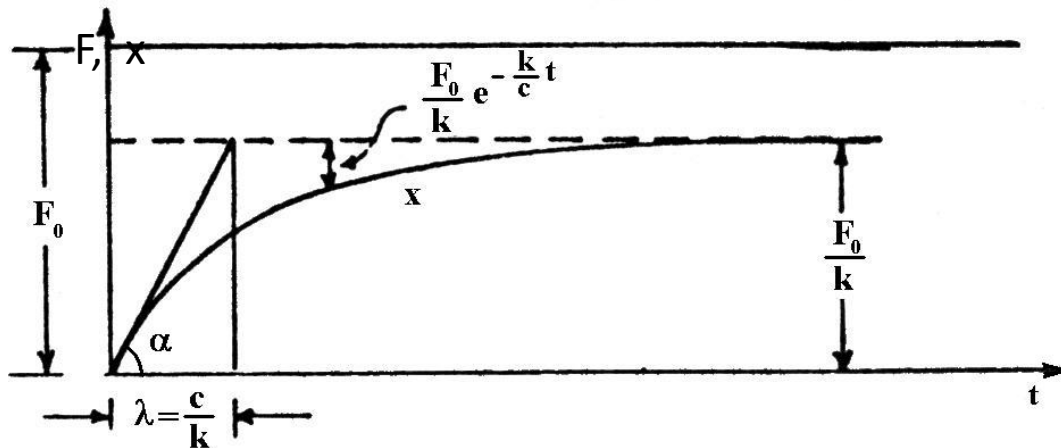
The 1st order differential equation that describes the indicator deflection therefore is: $c\dot{x} + kx = F_0$ and the general solution has the form $x(t) = x_{tr} + x_{rg}$

- x_{rg} is a *particular solution* and describes the *steady state condition* of the instrument:

Note that when $t \rightarrow \infty$ the indicator speed must be zero $\dot{x} = 0$, therefore the steady state of the indicator S will be described by the position $x_{rg} = F_0/k$

- x_{tr} is the *solution of the associated homogeneous equation* and describes the *dying out transient*: $x_{tr} = -\frac{F_0}{k} e^{-\frac{k}{c}t}$

The final solution will be the “sum” of the two contributions:
$$x(t) = x_{tr} + x_{rg} = -\frac{F_0}{k} e^{-\frac{k}{c}t} + \frac{F_0}{k} = \frac{F_0}{k} \left(1 - e^{-\frac{k}{c}t} \right)$$

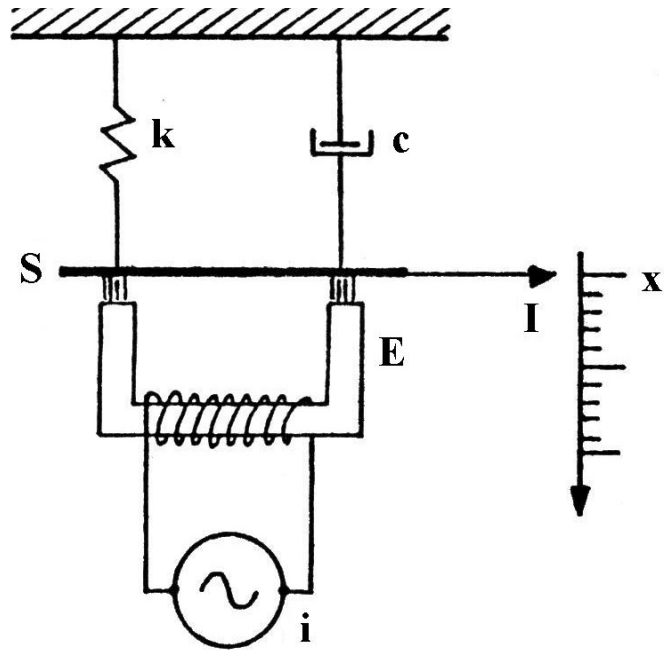


Observe again on the figure the meaning of $\lambda = \frac{c}{k}$

$$\operatorname{tg} \alpha = \frac{F_0/k}{\lambda} = \dot{x}(0)$$

At $t = \lambda$ we have $x(\lambda) = \frac{F_0}{k} (1 - e^{-1}) = 0.63 \frac{F_0}{k}$

When we are interested in studying the *steady state dynamic response* of a *1st order instrument* we employ the other tool, the **frequency response**. To do so, we have to apply at the instrument input a *periodic measurand*: $F(t) = F_0 \sin \omega t$ a periodic force of frequency $\omega = 2\pi f$!



The differential equation which describes the indicator S movements will be: $c\dot{x} + kx = F_0 \sin \omega t$

The general solution will always be $x(t) = x_{tr} + x_{rg}$ but we will be interested only in the steady state part, which we can assume to be:

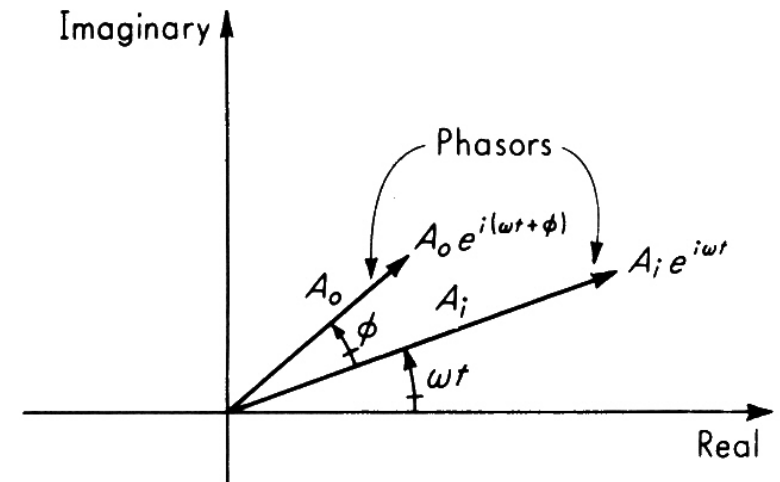
$$x_{rg}(t) = X_0 \sin(\omega t + \varphi)$$

How does the “amplitude” of the *output indication* $x(t)$ change with frequency ? And what “delay” will the *output* $x(t)$ show ?

Let's shift to the phasor notation:

$$\text{IN: } F(t) = F_0 \sin \omega t = F_0 e^{j\omega t}$$

$$\text{OUT: } x(t) = X_0 \sin(\omega t + \varphi) = X_0 e^{j\omega t} e^{j\varphi}$$



To verify the x_{rg} solution we have to substitute our assumed “steady state solution” in the differential equation ...

$$c\dot{x} + kx = F_0$$

$$j\omega c X_0 e^{j\omega t} e^{j\varphi} + k X_0 e^{j\omega t} e^{j\varphi} = F_0 e^{j\omega t}$$

$$X_0 e^{j\varphi} (j\omega c + k) = F_0$$

$$X_0 e^{j\varphi} = \frac{F_0}{j\omega c + k} = \frac{\frac{F_0}{k}}{j\omega \frac{c}{k} + 1} \quad \text{where } \frac{F_0}{k} \text{ is the } \underline{\text{maximum amplitude}} \text{ and } \frac{c}{k} = \lambda \text{ is the } \textbf{time constant} !$$

... calculating the modulus of the rationalized function, we will obtain:

$$X_0 e^{j\varphi} = \frac{F_0/k}{1 + j\omega\lambda} * \frac{1 - j\omega\lambda}{1 + j\omega\lambda} = \frac{F_0/k * (1 - j\omega\lambda)}{1 + (\omega\lambda)^2} = \underline{\frac{F_0/k}{1 + (\omega\lambda)^2}} - j \underline{\frac{F_0/k * \omega\lambda}{1 + (\omega\lambda)^2}}$$

$$| \quad | = \sqrt{\frac{\left(F_0/k\right)^2}{(1 + (\omega\lambda)^2)^2} + \frac{\left(F_0/k\right)^2 * (\omega\lambda)^2}{(1 + (\omega\lambda)^2)^2}} = F_0/k \sqrt{\frac{1 + (\omega\lambda)^2}{(1 + (\omega\lambda)^2)^2}} = \frac{F_0/k}{\sqrt{1 + (\omega\lambda)^2}}$$

$$X_0 = \frac{\frac{F_0}{k}}{\sqrt{(\omega\lambda)^2 + 1}}$$

$$\frac{X_0}{\frac{F_0}{k}} = G = \frac{1}{\sqrt{(\omega\lambda)^2 + 1}}$$

which is the **amplification** or **gain** of the instrument !

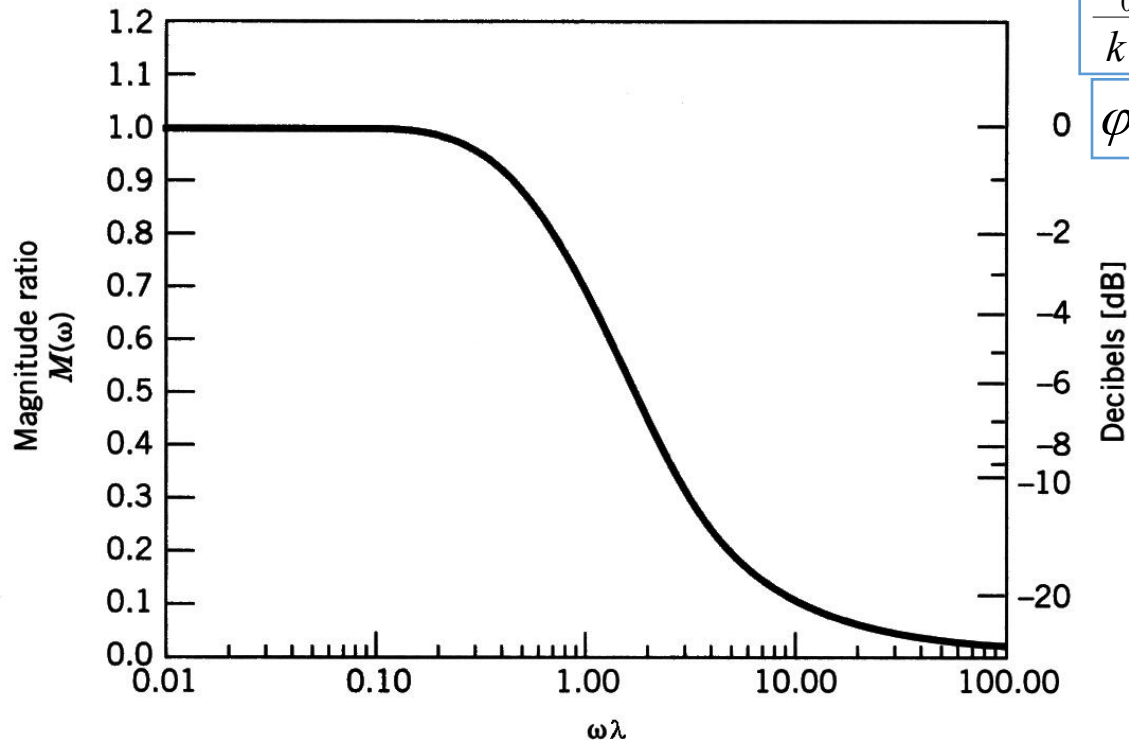
Output amplitude / maximum amplitude

$$\varphi = \arctg\left(\frac{Imm}{Re}\right)$$

$$\varphi = \arctg(-\omega\lambda)$$

which is the **phase delay** of the instrument !

1st order instrument frequency response

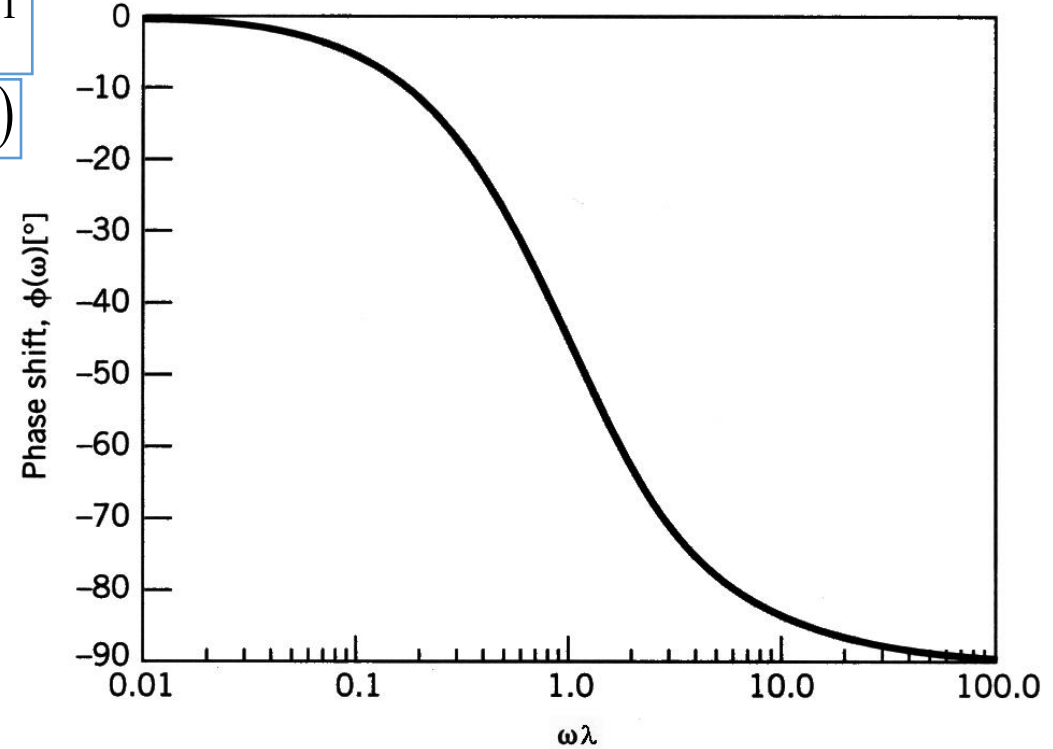


$$\frac{X_0}{F_0} = G = \frac{1}{\sqrt{(\omega\lambda)^2 + 1}}$$

$$\varphi = \arctg(-\omega\lambda)$$

$$\begin{aligned} \omega &\rightarrow 0 \\ \omega &\rightarrow \infty \\ \omega &= 1/\lambda \end{aligned}$$

1st order instrument phase delay



These diagram have on the x axis the non-dimensional variable “ $\omega\lambda$ » and therefore are normalized diagrams

$\omega_c = \frac{1}{\lambda}$ is the **characteristic frequency** of a 1st order instrument for which we have $\omega_c\lambda = 1 \rightarrow G = \frac{1}{\sqrt{2}} \cong 0.707$

therefore, ω_c is the **cut-off frequency** with a - 3db attenuation : $f_c = \frac{1}{2\pi \cdot \lambda}$

Example:

The “dynamic response” of the mercury thermometer

For every *differential time interval* dt that precedes the equilibrium condition we have:

Heat amount that the environment “transfers to” the mercury: $dQ = kA(T_a - T)dt$

Heat the mercury “receives from” the environment: $dQ = mc \cdot dT$

Where: m is the total mercury *mass*

c is the mercury *specific heat*

k is the *thermal exchange coefficient*

A is the *thermal exchange surface* of the thermometer

The two *heat amounts* of above must be the same:

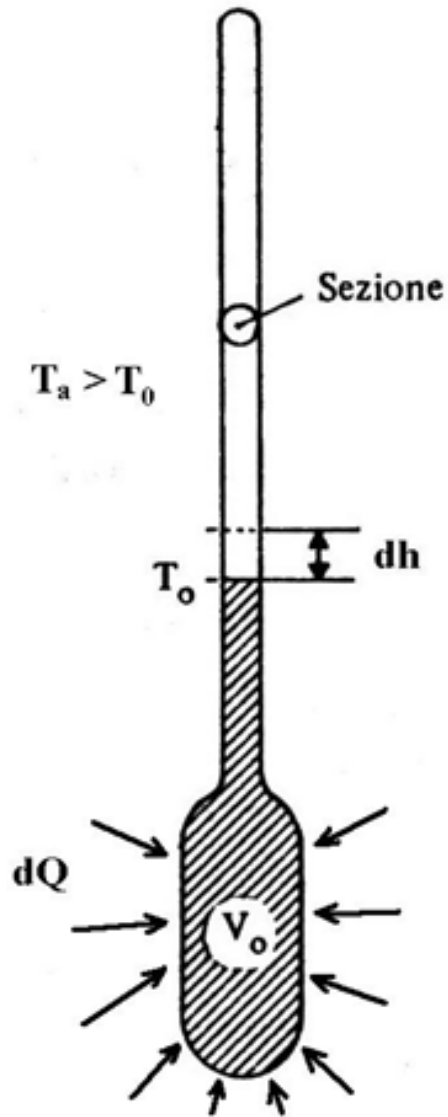
$$mc \cdot dT = kA(T_a - T)dt \quad mc \cdot \frac{dT}{dt} = kAT_a - kAT \quad \rightarrow \quad \frac{mc}{kA} \cdot \frac{dT}{dt} + T = T_a$$

Then the **time constant** for this **1st order instrument** is

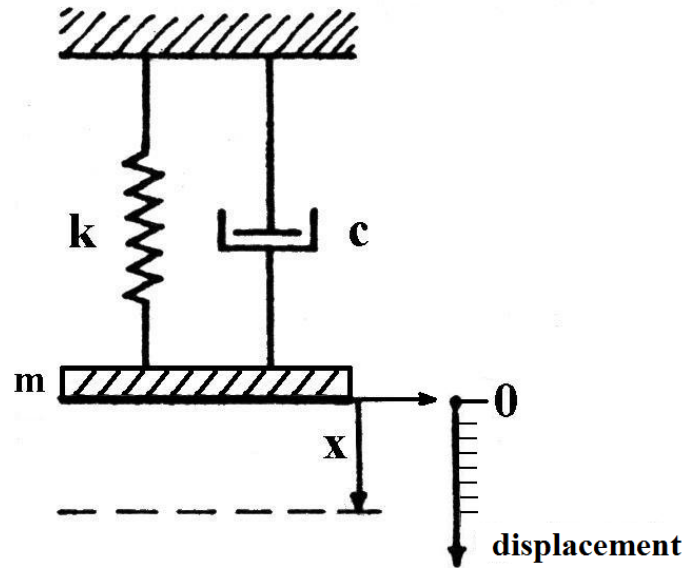
$$\lambda = \frac{mc}{kA}$$

$$c\dot{x} + kx = F_0$$

If $\lambda = 1s$ then $\omega\lambda = 1 \rightarrow \omega_c \cdot 1 = 1$ and $2\pi f_c = 1 \rightarrow f_c = \frac{1}{2\pi} = 0.16Hz$ quite slow !



Second-order instruments



A **second-order instrument** can “store energy” in two of its inner elements !

$k \rightarrow$ the elastic spring stores potential energy

$m \rightarrow$ the moving mass stores kinetic energy

The *differential equation* that describes the “free movement” of the indicator m

$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = 0$$

If we consider the very particular case for which $c = 0$ we get the important limit situation described by :

$$\frac{d^2 x}{dt^2} + \frac{k}{m} x = 0 \quad \text{where} \quad \boxed{\frac{k}{m} = \omega_n^2} \quad \text{and} \quad \omega_n \text{ is the } \textcolor{blue}{\text{natural frequency}} \text{ of the instrument !}$$

A system like this has NO damping and, if pulled from its equilibrium position $x = 0$, it will oscillate indefinitely with a frequency ω_n and will NEVER stop !

Of course, it's an IDEAL system ...

In the real case of the **step response**, an *input variable* is present : a step force F_0 is applied to the indicator m at time $t = 0 \dots$

The 2nd order differential equation that describes the indicator deflection therefore is: $m\ddot{x} + c\dot{x} + kx = F_0$ and the general solution has again the form $x(t) = x_{tr} + x_{rg}$

- x_{rg} is a *particular solution* and describes the *steady state condition* of the instrument:

Note that when $t \rightarrow \infty$ the indicator speed and acceleration must be zero $\ddot{x} = \dot{x} = 0$ therefore the steady state of the indicator S will be described by the position $x_{rg} = F_0/k$

- x_{tr} is the solution of the *associated homogeneous equation* and describes the *dying out transient*:

$$x_{tr}(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \quad \text{with} \quad \alpha_{1,2} = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}} \quad \text{and} \quad \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}} = \sqrt{\Delta} \quad \text{the **discriminant** !}$$

- Depending on the sign of the discriminant $\Delta > 0$, $\Delta = 0$, $\Delta < 0$ we have “different behaviors” of the instrument
- The 2nd order instrument will exhibit “more than one type” of step response !
- The curves of the step response will depend on a parameter obtained from the discriminant Δ :

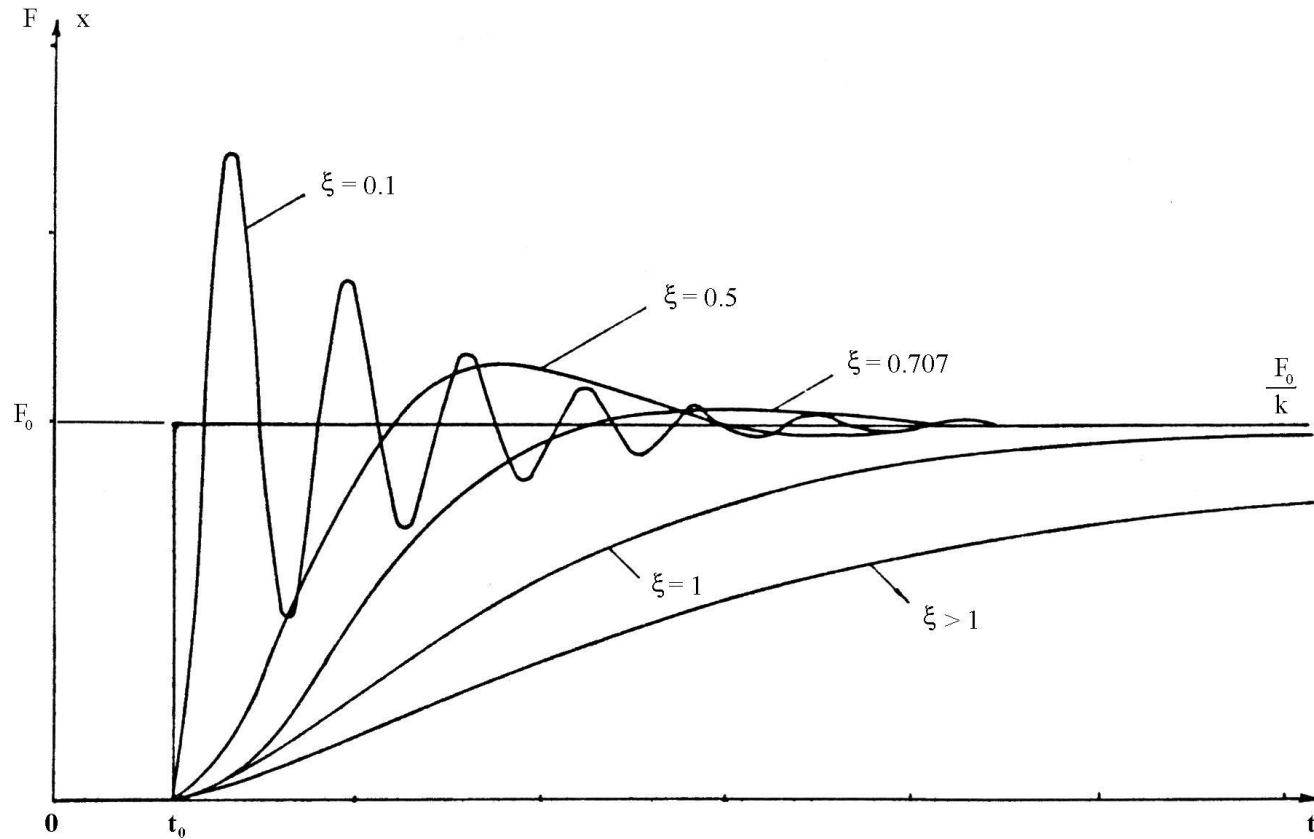
$$\xi = \sqrt{\frac{\frac{c^2}{4m^2}}{\frac{k}{m}}} = \sqrt{\frac{c^2}{4km}} = \frac{c}{2\sqrt{km}}$$

the damping factor

$$\xi = \frac{c}{c_{cr}}$$

where

$$c_{cr} = 2\sqrt{km}$$



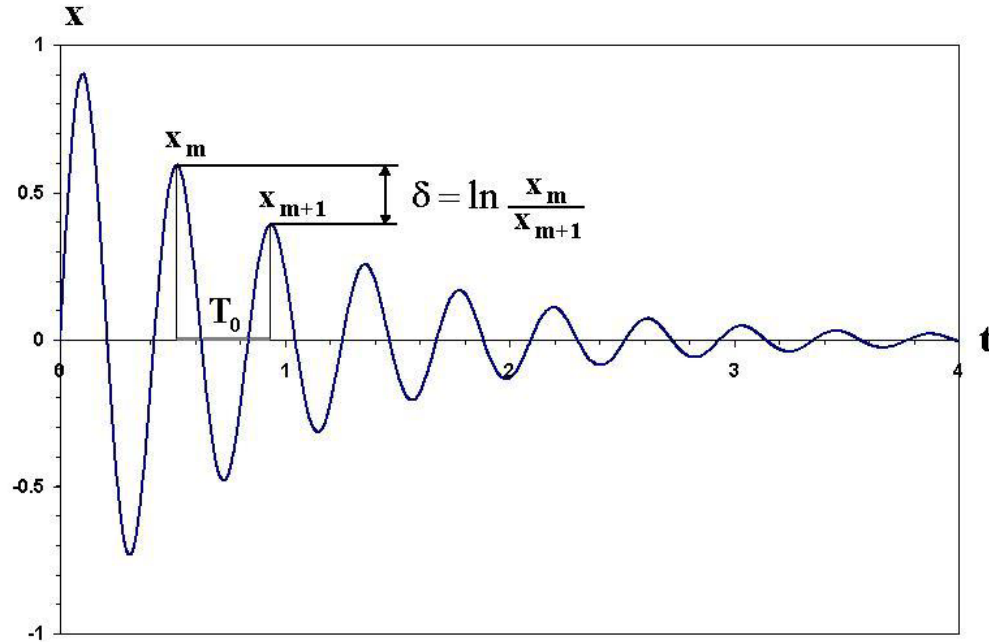
- For $\xi > 1$ (which is the same of $c > c_{cr}$) the instrument is over-damped and it will reach the final indication with a slow exponential path !
- For $\xi = 1$ (which is $c = c_{cr}$) the instrument has a critical damping and reaches the final indication with the fastest exponential path !
- For $\xi < 1$ (which is $c < c_{cr}$) the instrument is under-damped, it reacts quickly to the step input but exhibits several damped *over-oscillations* before reaching the final indication !

Note that for the case $\xi < 1$ the **frequency** of the over-oscillations seems to depend from the value of the damping factor ξ ...

Instrument designer generally prefer to choose a damping factor $\xi = 0.7 \div 0.8$ so the indicator ***m*** will have only one over-oscillation and reaches the final indication F_0/k from the upper side ...

To obtain the values of ω_n and ξ one should know the values of the coefficients m , c , and k of the 2nd order instrument which is often almost impossible !

In case the instrument has $\xi < 1$ (*which is the most common*) one can employ an experimental procedure that leads to the **logarithmic decay δ** :



Displace the instrument indicator from the zero position, leave it free to move back and measure the decreasing oscillations ...

$$\delta = \ln \frac{x_m}{x_{m+1}} = \frac{1}{k} \ln \frac{x_m}{x_{m+k}}$$

If the decay is small, it can also be measured “after k waves”, as showed in the equation above.

The *logarithmic decay δ* is dependent on the *damping factor ξ* :

$$\delta = \ln \frac{A_n}{A_{n+1}} = 2\pi \frac{\xi}{\sqrt{1-\xi^2}}$$

And the frequency $\omega_0 = 2\pi/T_0$ of the decreasing oscillation is “smaller” than the natural frequency ω_n ... depending also from the damping factor ξ :

$$\omega_0 = \omega_n \sqrt{1-\xi^2}$$

When we are interested in studying the *steady state dynamic response* of a *2nd order instrument* we have to employ the other tool, the **frequency response**. To do so, we have to apply at the instrument input a *periodic measurand*: $F(t) = F_0 \sin \omega t$ a periodic force of frequency $\omega = 2\pi f$!

The differential equation which describes the indicator ***m*** movements will be: $m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t$

The general solution will always be $x(t) = x_{tr} + x_{rg}$ but we will be interested only in the steady state part, which we can assume to be: $x_{rg}(t) = X_0 \sin(\omega t + \varphi)$

How does the “amplitude” X_0 of the *output indication* $x(t)$ change with frequency ?

And what “phase delay” ϕ will the *output* $x(t)$ show ?

As previously done for the 1st order instruments we substitute the input $F(t) = F_0 e^{j\omega t}$ and the instrument output $x(t) = X_0 e^{j\omega t} e^{j\varphi}$ in the differential equation:

$$m(-\omega^2)X_0 e^{j\omega t} e^{j\varphi} + c(j\omega)X_0 e^{j\omega t} e^{j\varphi} + kX_0 e^{j\omega t} e^{j\varphi} = F_0 e^{j\omega t}$$

$$X_0 e^{j\varphi} (-m\omega^2 + jc\omega + k) = F_0 \quad X_0 e^{j\varphi} = \frac{F_0}{-m\omega^2 + jc\omega + k} = \frac{\frac{F_0}{k}}{-\frac{m}{k}\omega^2 + j\frac{c}{k}\omega + 1}$$

$$X_0 e^{j\varphi} = \frac{\frac{F_0}{k}}{-\frac{\omega^2}{\omega_n^2} + j2\xi \frac{\omega}{\omega_n} + 1} = \frac{\frac{F_0}{k}}{1 - \frac{\omega^2}{\omega_n^2} + j2\xi \frac{\omega}{\omega_n}}$$

with

$$\frac{k}{m} = \omega_n^2$$

$$j \frac{c}{k} \omega = j \frac{\xi c_{cr}}{k} \omega = j \frac{\xi 2\sqrt{km}}{k} \omega = j 2\xi \sqrt{\frac{m}{k}} \omega = j 2\xi \frac{\omega}{\omega_n}$$

$$X_0 e^{j\varphi} = \frac{F_0}{-m\omega^2 + jc\omega + k} = \frac{\frac{F_0}{k}}{-\frac{m}{k}\omega^2 + j\frac{c}{k}\omega + 1}$$

which is the complex frequency response for the 2nd order instrument

Calculating the *modulus* and the *phase lag* of this function we obtain ...

The **amplitude** of the **frequency response** of the instrument:

$$X_0 = \frac{\frac{F_0}{k}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \left(\frac{\omega}{\omega_n}\right)^2}}$$

$$\xi : 0.707$$

Normalized diagram of the **amplification** or **gain** of the instrument:

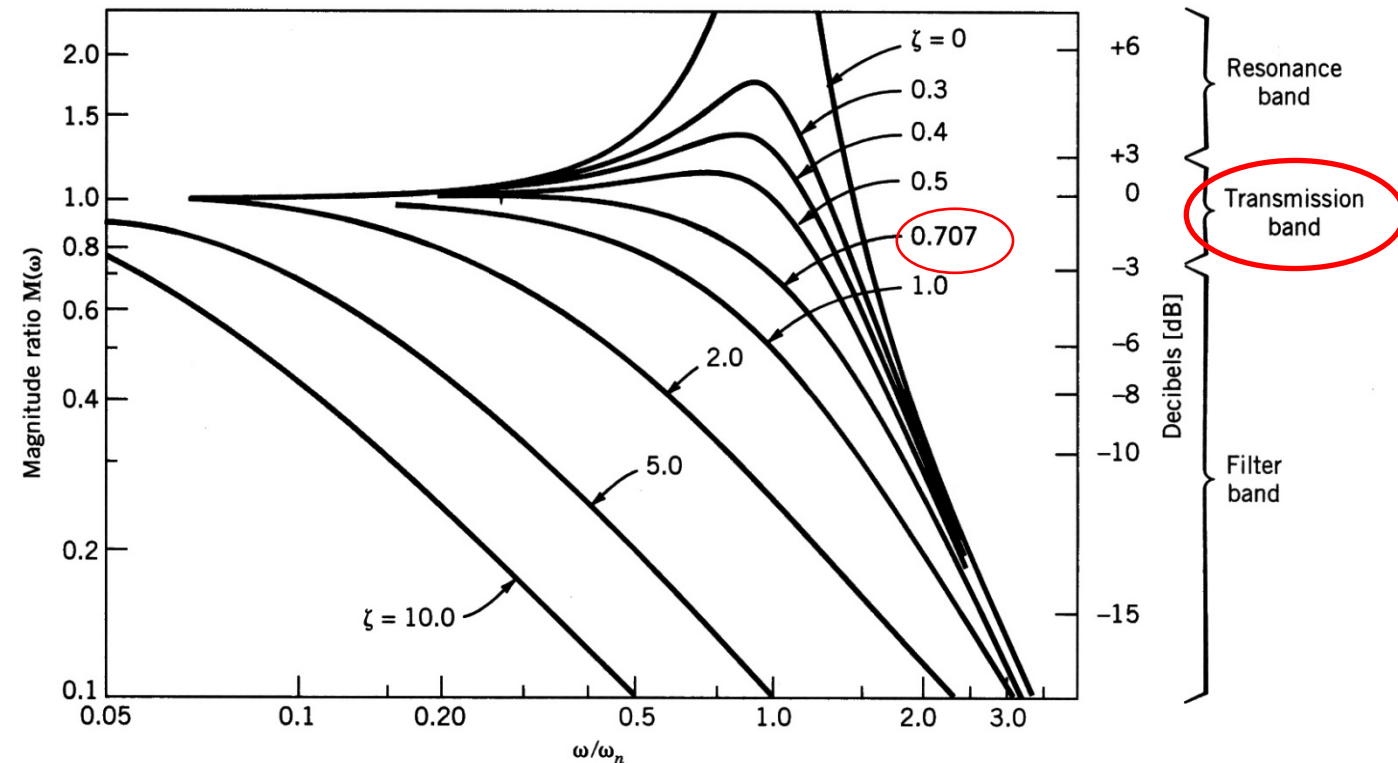
$$G = \frac{X_0}{F_0 / k}$$

$$G = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \left(\frac{\omega}{\omega_n}\right)^2}}$$

These curves are also dependent from the damping factor ξ

For $\frac{\omega}{\omega_n} = 1$ the amplification is: $G = \frac{1}{2\xi}$

wich for «small values» of ξ can be «dangerous», these are the **resonance conditions** ...



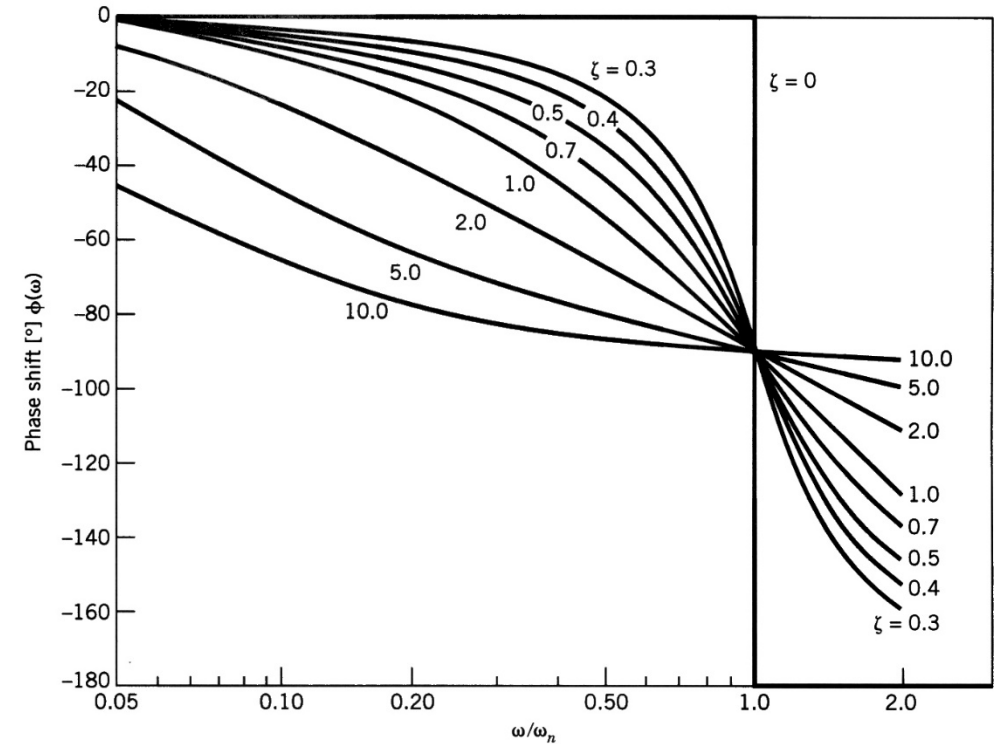
$$\varphi = \arctg \frac{-2\xi \frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

the **phase delay** of the instrument output !

The curves depend on the damping factor ξ , however they all pass through the point :

$$\frac{\omega}{\omega_n} = 1 \quad \varphi = -\frac{\pi}{2}$$

which makes it useful to experimentally determine the natural frequency ω_n without knowing the damping factor ξ ...

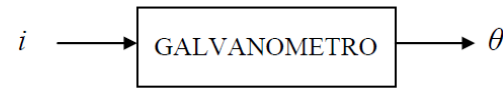


If you consider $x_{rg}(t) = \frac{F_0}{k}$ the steady state output for the step response, the static sensitivity will result:

$$S = \frac{du}{di} = \frac{dx}{dF} = \frac{1}{k} = \frac{1}{m\omega_n^2}$$

which makes the **dynamic response** of an instrument always inversely proportional to its **static sensitivity** !!

Example: the galvanometer



Static characteristic (graduation curve)

$$\vec{F} = i\vec{l} \times \vec{B} \quad |F| = ilB$$

motor torque: $C_m = nF \cdot b$

resisting torque: $C_r = k \cdot \theta$

$$C_m = C_r \quad n \cdot ilB \cdot b = k\theta$$

$$\theta = \frac{nIBb}{k} \cdot i$$

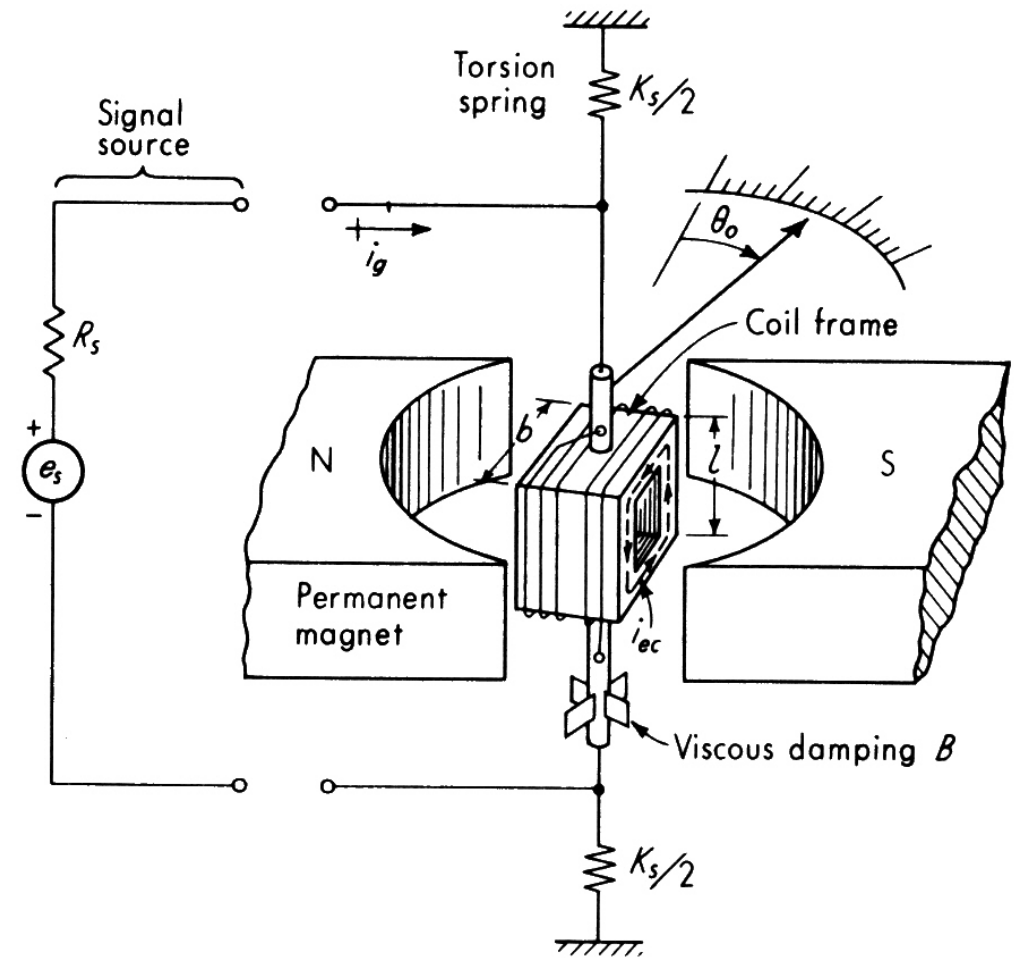
$$S = \frac{d\theta}{di} = \frac{nIBb}{k} = \frac{nIBb}{J\omega_n^2}$$

the sensitivity is constant and inversely proportional to ω_n

Dynamic characteristic (dynamic response)

The output indication is a rotation θ therefore : $\omega_n = \sqrt{\frac{k}{J}}$

$$J \frac{d^2\theta}{dt^2} + c \frac{d\theta}{dt} + k\theta = C(t)$$



and $C(t) = C_0$ for $t > 0$ (step response)

$C(t) = C_0 \sin \omega t$ (frequency response)