

Risoluzione di forme indeterminate.

$$\lim_{n \rightarrow +\infty} (\underbrace{\sqrt{n^3+n}}_{+\infty} - \underbrace{n}_{+\infty}) = (+\infty - \infty) =$$

$$= \lim_{n \rightarrow +\infty} (n^{3/2} \underbrace{\sqrt{1+\frac{1}{n^2}}}_{\substack{\downarrow \\ \sqrt{1} \\ = 1}} - n) = \lim_{n \rightarrow +\infty} \underbrace{n^{3/2}}_{+\infty} \left(\underbrace{\sqrt{1+\frac{1}{n^2}}}_{\substack{\downarrow \\ \sqrt{1} \\ = 1}} - \underbrace{\frac{1}{\sqrt{n}}}_0 \right) =$$
$$= (+\infty \cdot 1) = +\infty$$

$$\lim_{n \rightarrow +\infty} (\sqrt{n^2+5} - 2n) = (+\infty - \infty) =$$

$$= \lim_{n \rightarrow +\infty} (n \sqrt{1+\frac{5}{n^2}} - 2n) = \lim_{n \rightarrow +\infty} \underbrace{n}_{+\infty} \left(\underbrace{\sqrt{1+\frac{5}{n^2}}}_{\substack{\downarrow \\ 1}} - \underbrace{2}_{-2} \right) = (+\infty \cdot (-1))$$
$$= -\infty$$

$$\lim_{n \rightarrow +\infty} (\sqrt{n^2+5} - n) = (+\infty - \infty) =$$

$$= \lim_{n \rightarrow +\infty} \underbrace{n}_{+\infty} \left(\underbrace{\sqrt{1+\frac{5}{n^2}} - 1}_0 \right) = (+\infty \cdot 0) \text{ (non si conclude)}$$

Allora sfruttiamo un differente approccio. Ricordiamo che:

$$(A-B)(A+B) = A^2 - B^2, \text{ e prendiamo}$$

$$A = \sqrt{n^2+5}, \quad B = n$$

↙ moltiplico e divido

$$\sqrt{n^2+5} - n = \frac{(\sqrt{n^2+5} - n)(\sqrt{n^2+5} + n)}{\sqrt{n^2+5} + n} =$$

$$= \frac{\cancel{n^2+5} - \cancel{n^2}}{\underbrace{\sqrt{n^2+5} + n}_{\downarrow +\infty}} \rightarrow \left(\frac{5}{+\infty} \right) = 0$$

$$\lim_{n \rightarrow +\infty} \left(\underbrace{\sqrt[3]{n^3+5}}_{n \cdot \sqrt[3]{1+\frac{5}{n^3}}} - n \right) = (+\infty - \infty) =$$

$$(A-B)(A^2+AB+B^2) = A^3 - B^3$$

$$= \lim_{n \rightarrow +\infty} \frac{(\sqrt[3]{n^3+5} - n) \left((n^3+5)^{2/3} + n \sqrt[3]{n^3+5} + n^2 \right)}{(n^3+5)^{2/3} + n \sqrt[3]{n^3+5} + n^2} =$$

$$= \lim_{n \rightarrow +\infty} \frac{\cancel{n^3+5} - \cancel{n^3}}{\underbrace{(n^3+5)^{2/3}}_{\downarrow +\infty} + \underbrace{n \sqrt[3]{n^3+5}}_{\downarrow +\infty} + \underbrace{n^2}_{\downarrow +\infty}} = \left(\frac{5}{+\infty} \right) = 0$$

$$\lim_{n \rightarrow +\infty} \left(\sqrt[3]{n^3 + n^2} - n \right) = (+\infty - \infty) =$$

$$= \lim_{n \rightarrow +\infty} \frac{\left(\sqrt[3]{n^3 + n^2} - n \right) \left((n^3 + n^2)^{2/3} + n \sqrt[3]{n^3 + n^2} + n^2 \right)}{(n^3 + n^2)^{2/3} + n \sqrt[3]{n^3 + n^2} + n^2} =$$

$$= \lim_{n \rightarrow +\infty} \frac{\cancel{n^3} + n^2 - \cancel{n^3}}{(n^3 + n^2)^{2/3} + n \sqrt[3]{n^3 + n^2} + n^2} =$$

$$= \lim_{n \rightarrow +\infty} \frac{\cancel{n^2} \quad 1}{\cancel{n^2} \left[\left(1 + \frac{1}{n} \right)^{2/3} + \sqrt[3]{1 + \frac{1}{n}} + 1 \right]} = \frac{1}{3}$$

$$\left((n^3 + n^2)^{2/3} = n^2 \left(1 + \frac{1}{n} \right)^{2/3} \right)$$

$$\lim_{n \rightarrow +\infty} \left(4 - \frac{1}{2n} \right)^{n-5} = +\infty$$

Oss si tratta di una successione del tipo
 $(a_n)^{b_n}$ con $\underline{a_n > 0}$

$$\forall x > 0$$

$$x = 2^{\log_2 x}$$

$$\forall x > 0$$

(potrei anche usare una base diversa)

$$(a_n)^{b_n} = 2^{\log_2(a_n)^{b_n}} = 2^{b_n \log_2(a_n)}$$

$$\left(4 - \frac{1}{2n}\right)^{n-5} = 2^{(n-5) \log_2 \left(4 - \frac{1}{2n}\right)}$$

Mi concentro sull'esponente.

$$\underbrace{(n-5)}_{\downarrow +\infty} \underbrace{\log_2 \left(4 - \frac{1}{2n}\right)}_{\downarrow \log_2 4 = 2} \rightarrow +\infty$$

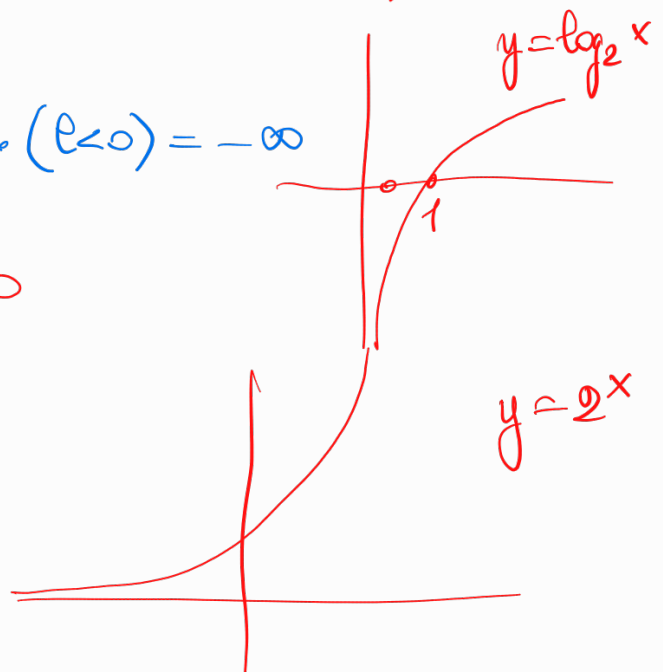
quindi $2^{\boxed{(n-5) \log_2 \left(4 - \frac{1}{2n}\right)}} \rightarrow +\infty$

$$\lim_{n \rightarrow +\infty} \left(\frac{1}{3} + \frac{1}{2n}\right)^{n-5}$$

$$\left(\frac{1}{3} + \frac{1}{2n}\right)^{n-5} = 2^{\boxed{(n-5) \log_2 \left(\frac{1}{3} + \frac{1}{2n}\right)}} \rightarrow \boxed{\rightarrow -\infty} \rightarrow (2^{-\infty}) = 0$$

$$\underbrace{(n-5)}_{\downarrow +\infty} \underbrace{\log_2 \left(\frac{1}{3} + \frac{1}{2n}\right)}_{\downarrow \log_2 \left(\frac{1}{3}\right) = -\log_2 3 < 0} \rightarrow (+\infty) \cdot (< 0) = -\infty$$

$$\log_2 \left(\frac{1}{3}\right) = -\log_2 3 < 0$$



Consideriamo la successione $(a_n)^{b_n}$, con $a_n > 0$

se $b_n \rightarrow +\infty$ e $a_n \rightarrow l \in (0, +\infty)$

$$(a_n)^{b_n} = 2^{b_n \log_2(a_n)} \rightarrow \begin{cases} (2^{+\infty}) = +\infty & \text{se } l > 1 \\ (2^{-\infty}) = 0 & \text{se } l < 1 \end{cases}$$

$$\begin{array}{ccc} b_n \log_2(a_n) & \rightarrow & \begin{cases} +\infty & \text{se } l > 1 \\ -\infty & \text{se } l < 1 \end{cases} \\ \downarrow \quad \downarrow & & \\ +\infty \quad \log_2 l & & \\ & & \begin{array}{l} > 0 \text{ se } l > 1 \\ < 0 \text{ se } l < 1 \end{array} \end{array}$$

se $l = +\infty$ ($a_n \rightarrow +\infty$) $b_n \rightarrow +\infty$, il limite vale $+\infty$
 $a_n \rightarrow l = 0$, $b_n \rightarrow +\infty$, il limite vale 0

Se $a_n \rightarrow 1$, $b_n \rightarrow +\infty$

$$(a_n)^{b_n} = 2^{\underbrace{b_n}_{\rightarrow +\infty} \underbrace{\log a_n}_{\rightarrow 0}}$$

Ricapitoliamo: Sia $a_n > 0$

se $a_n \rightarrow l$, $b_n \rightarrow +\infty$

$$(a_n)^{b_n} \rightarrow \begin{cases} +\infty & \text{se } l \in (1, +\infty] \\ 0 & \text{se } l \in [0, 1) \end{cases}$$

Riassunto veloce

$$"l^{+\infty} = +\infty" \quad \forall l > 1$$

$$"(+\infty)^{+\infty} = +\infty"$$

$$\begin{aligned} & "l^{+\infty} = 0 \quad \forall l \in [0, 1) \\ & "1^{+\infty} \text{ è una f.i."} \end{aligned}$$

Altro modo

$$\left(4 - \frac{1}{2n}\right)^{n-5} \stackrel{\text{def}^{\text{te}}}{>} 3^{n-5} \rightarrow +\infty$$

$$4 - \frac{1}{2n} \rightarrow 4 \quad \text{def}^{\text{te}}$$

$$4 - \frac{1}{2n} > 3 \Rightarrow$$

$$\Rightarrow \text{def}^{\text{te}}$$

$$\left(4 - \frac{1}{2n}\right)^{n-5} > 3^{n-5} \rightarrow +\infty$$

quindi (thm. carabinieri)

$$\left(4 - \frac{1}{2n}\right)^{n-5} \rightarrow +\infty$$

$$\left(\frac{1}{3} + \frac{1}{2n}\right)^{n-5}$$

$$\stackrel{\text{def}^{\text{te}}}{\frac{1}{3} + \frac{1}{2n} < \frac{1}{2}} \Rightarrow \left(\frac{1}{3} + \frac{1}{2n}\right)^{n-5} < \left(\frac{1}{2}\right)^{n-5}$$

$0 < \quad \downarrow \quad 0$

$\Rightarrow$ thm. carab.

$$\left(\frac{1}{3} + \frac{1}{2n}\right)^{n-5} \rightarrow 0$$

$$\text{se } \boxed{a_n \rightarrow l, \quad b_n \rightarrow -\infty}$$

$$(a_n)^{b_n} = 2^{\underbrace{b_n}_{\downarrow -\infty} \underbrace{\log_2 a_n}_{\downarrow \log_2 l}}$$

$\rightarrow -\infty$ se $l > 1$
 $\rightarrow +\infty$ se $0 < l < 1$
 > 0 se $l > 1$
 < 0 se $0 < l < 1$

$$\Rightarrow (a_n)^{b_n} \rightarrow \begin{cases} 0 & \text{se } l > 1, \text{ opp } l = +\infty \\ +\infty & \text{se } 0 \leq l < 1 \end{cases}$$

$$l^{-\infty} = \begin{cases} 0 & \text{se } l > 1 \\ +\infty & \text{se } 0 \leq l < 1 \end{cases}$$

$$(+\infty)^{-\infty} = 0$$

" $1^{-\infty}$ " è una f.i.

Se $a_n \rightarrow l \in \mathbb{R}^+$ $b_n \rightarrow m \in \mathbb{R}$

$$(a_n)^{b_n} = 2^{\underbrace{b_n \log_2 a_n}_{m \log_2 l}} \rightarrow 2^{m \log_2 l} = l^m$$

Se $a_n \rightarrow +\infty$ $b_n \rightarrow m \in \mathbb{R}$.

$$(a_n)^{b_n} = 2^{\underbrace{b_n \log_2 a_n}_{\begin{cases} +\infty & \text{se } m > 0 \\ -\infty & \text{se } m < 0 \end{cases}}} \rightarrow \begin{cases} +\infty & \text{se } m > 0 \\ 0 & \text{se } m < 0 \end{cases}$$

$$(+\infty)^m = \begin{cases} +\infty & \text{se } m > 0 \\ 0 & \text{se } m < 0 \end{cases}$$

" $(+\infty)^0$ " è una f.i.

se $a_n \rightarrow 0^+$, $b_n \rightarrow m$

$$\Rightarrow (a_n)^{b_n} = 2^{\underbrace{b_n \log_2 a_n}_{-\infty}} \rightarrow \begin{cases} +\infty & \text{se } m < 0 \\ 0 & \text{se } m > 0 \end{cases}$$

$$0^m = \begin{cases} +\infty & \text{se } m < 0 \\ 0 & \text{se } m > 0 \end{cases}$$

" 0^0 " è una f.i.

Abbiamo trovato 4 f.i. nuove

$$1 \pm \infty \quad (+\infty)^0 \quad 0^0$$

$$\lim_{n \rightarrow +\infty} \left(\frac{n^2 - 2n}{2n^2 + 1} \right)^{-n} = \left(\left(\frac{1}{2} \right)^{-\infty} \right) = +\infty$$

$$\left(\frac{n^2 - 2n}{2n^2 + 1} \right)^{-n} \rightarrow \frac{1}{2}$$

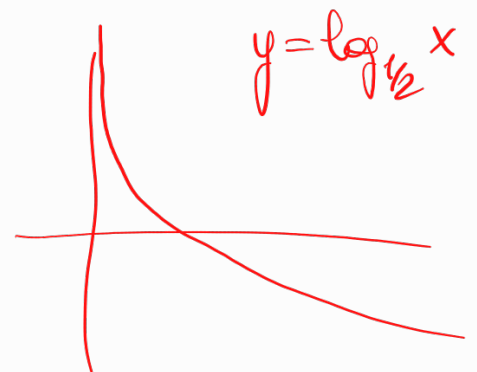
$$\log_2 \frac{1}{2} = -1$$

$$\left(\frac{n^2 - 2n}{2n^2 + 1} \right)^{-n} = 2^{\underbrace{-n \log_2 \left(\frac{n^2 - 2n}{2n^2 + 1} \right)}_{+\infty}} \rightarrow (2^{+\infty}) = +\infty$$

$$\lim_{n \rightarrow +\infty} \left[\underbrace{\log_{1/2}(n^3)}_{-\infty} - \underbrace{\log_{1/2}(n+1)}_{-\infty} \right] =$$

$$= (-\infty + \infty) =$$

$$= \lim_{n \rightarrow +\infty} \log_{1/2} \left(\frac{n^3}{n+1} \right) = -\infty$$



TEOREMA (limite di succ^{ie} monotone, cioè crescenti o decrescenti)

Sia $\{a_n\}$ una succ^{ie} crescente di numeri reali.
decrescente

(Parentesi: a_n è crescente se $\forall n_1, n_2 \in \mathbb{N}$ t.c. $n_1 < n_2$
 $a_{n_1} \leq a_{n_2}$, o equivalentemente se
 $a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}.$)

Allora $\exists \lim_{n \rightarrow +\infty} a_n = \sup_{n \in \mathbb{N}} a_n$
 $\inf_{n \in \mathbb{N}} a_n$

Applicazione:

Cerchiamo $\sup$ e $\inf$ di $a_n = \frac{n^2 - n + 1}{n^2 + 1}, n \in \mathbb{N}_+$

Vediamo se a_n è crescente/decrescente.

$$a_{n+1} \stackrel{?}{>} a_n \Leftrightarrow \frac{(n+1)^2 - (n+1) + 1}{(n+1)^2 + 1} \stackrel{?}{>} \frac{n^2 - n + 1}{n^2 + 1}$$

$$\Leftrightarrow \frac{n^2 + 2n + 1 - n - 1 + 1}{n^2 + 2n + 2} \stackrel{?}{>} \frac{n^2 - n + 1}{n^2 + 1}$$

$$\Leftrightarrow (n^2 + n + 1)(n^2 + 1) \stackrel{?}{>} (n^2 - n + 1)(n^2 + 2n + 2)$$

$$\Leftrightarrow \cancel{n^4} + \cancel{n^2} + \cancel{n^3} + \cancel{n} + \cancel{n^2} + 1 \stackrel{?}{>} \cancel{n^4} + \cancel{2n^3} + \cancel{2n^2} - \cancel{n^3} - \cancel{2n^2} - \cancel{2n} + \cancel{n^2} + \cancel{2n} + 2$$

$$\Leftrightarrow n^2 + n - 1 > 0 \Leftrightarrow n > \frac{-1 + \sqrt{5}}{2} \sim 0,6$$

$$x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

sempre crescente per $n \geq 1$.

teor.

$$\Rightarrow \sup a_n = \lim_{n \rightarrow +\infty} a_n = \lim_{n \rightarrow +\infty} \frac{n^2 - n + 1}{n^2 + 1} = 1$$

$$\inf a_n = \min a_n = a_1 = \frac{1-1+1}{2} = \frac{1}{2}$$