Lecture 8-9
Exponential & Logarithmic Equations and Inequalities

Preliminary course - Mathematics
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Lecture 8-9

1. Exponential equations
2. Logarithmic equations
3. Exponential inequalities
4. Logarithmic inequalities

Exponential equations

An exponential equation involves variables in the exponent

$$a^x = b$$

Example: $2^x = 8$

- $a > 0$ is the base, $x \in \mathbb{R}$ is the unknown $\rightarrow$ **There are not existence conditions.**
- For every $x \in \mathbb{R}$, $a^x > 0$. For this reason $b > 0$.
- Properties of powers:
  - $a^m \cdot a^n = a^{m+n}$
  - $\frac{a^m}{a^n} = a^{m-n}$
  - $(a^m)^n = a^{mn}$
  - $a^0 = 1$
  - $a^{-x} = \frac{1}{a^x}$

Resolution

In order to solve an exponential equation, we need to have the **same base**:

$$a^{A(x)} = a^{B(x)} \Longrightarrow A(x) = B(x)$$

or we can use the inverse function of the exponential that is the logarithm (we'll see it further on).

Example: Solve $2^x = 8$

1. Express 8 as a power of 2: $8 = 2^3$.
2. Rewrite the equation: $2^x = 2^3$.
3. Since the bases are equal, set the exponents equal: $x = 3$.
4. **Solution:** $x = 3$.

Example

Solve $3^{2x} = 9^x$

1. Express 9 as a power of 3: $9 = 3^2$.
2. Rewrite the equation: $3^{2x} = (3^2)^x$.
3. Simplify: $3^{2x} = 3^{2x}$.
4. Since the bases are equal, set the exponents equal: $2x = 2x$.
5. **Solution:** All x are solutions.

Solve $(\frac{1}{3})^x = 27$

- Express 27 as a power of 3: $27 = 3^3$.
- Rewrite $\frac{1}{3}$ as 3^{-1}: $(3^{-1})^x = 3^3$.
- Simplify: $3^{-x} = 3^3$.
- Since the bases are equal, set the exponents equal: $-x = 3$.
- **Solution:** $x = -3$.

Exercises

1. $25^x = 125$
2. $64\sqrt{2} = 4^{2x}$
3. $5 \cdot 2^x + 2^x \cdot 2^{-2} = 336$
4. $3^x = \frac{\sqrt{3}}{9}$

Logarithmic Equations

The logarithm is the inverse function of the exponential:

$$\log_a(x) = b$$

The logarithm of x is the exponent to which a given base a must be raised to produce that number, i.e., $a^b = x$

- $a > 0$ is the base, $x > 0$ is the argument. The logarithm can assume all values, that is $b \in \mathbb{R}$.
- **Existence conditions:** $x > 0$

Properties:

1. $\log_a(a) = 1$
2. **Product Rule:** $\log_a(xy) = \log_a(x) + \log_a(y)$
3. **Quotient Rule:** $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
4. **Power Rule:** $\log_a(x^y) = y\log_a(x)$
5. **Change of Base Formula:** $\log_a(x) = \frac{\log_k(x)}{\log_k(a)}$
6. **Base 10 and Natural Logarithms:**
   - Common logarithm: $\log(x) = \log_{10}(x)$
   - Natural logarithm: $\ln(x) = \log_e(x)$
7. **Inverse function:** $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$
8. $\log_a(A(x)) = \log_a(B(x)) \Rightarrow A(x) = B(x)$

Resolution: Basic Logarithmic Equations I

Example 1: Solve $\log_3(x) = -1$

1. CE: $x > 0$
2. Rewrite the equation in exponential form: $3^{-1} = x$.
3. Solution of the equation: $x = \frac{1}{3}$
4. Check the EC: OK

Resolution: Basic Logarithmic Equations II

Example 2: Solve $\log_2(x^2 - 1) = \log_5(25)$

1. CE: $x^2 - 1 > 0$, i.e., $x < -1$ and $x > 1$
2. Rewrite $\log_5(25)$ using the power rule: $\log_5(25) = \log_5(5^2) = 2$.
3. The equation becomes: $\log_2(x^2 - 1) = 2$.
4. You can rewrite 2 as a logarithm, that is $2 = \log_2(4)$
5. We have
$$\log_2(x^2 - 1) = \log_2(4) \Rightarrow x^2 - 1 = 4$$
6. Solutions of the equation: $x = \pm\sqrt{5}$
7. Check the EC: OK

Solving Logarithmic equations using Exponential I

Example 3: Solve $\log(x) + \log(x-3) = 1$

1. CE: $x > 0$ and $x > 3$. Their intersection is $x > 3$.
2. Use the product rule: $\log(x(x-3)) = 1$.
3. Apply the exponential with base 10 to both sides: $10^{\log(x(x-3))} = 10^1$.
4. $x(x-3) = 10$, that is $x^2 - 3x - 10 = 0$
5. Solutions of the quadratic equation: $x = 5$ and $x = -2$.
6. Check the EC: NO. $x = -2$ cannot be a solution. **Solution:** $x = 5$.

Example 4: Solve $\log_2(x^2) = -1$

1. CE: $x^2 > 0$, for all $x \in \mathbb{R}$
2. Apply the exponential with base 2 to both sides: $2^{\log_2(x^2)} = 2^{-1}$.
3. $x^2 = \frac{1}{2}$.
4. **Solutions**: $x = \pm\sqrt{1/2} = \pm\frac{1}{\sqrt{2}}$.

Solving Logarithmic equations using Exponential II

Example 5: Solve $\ln\left(\frac{x}{x+1}\right) = 2$

1. CE: $\frac{x}{x+1} > 0$. Numerator: $x > 0$; Denominator: $x > -1$. The fraction is positive for $x < -1$ or $x > 0$.
2. Apply the exponential with base e to both sides and get
$$\frac{x}{x+1} = e^2.$$
3. Solve it as a fractional equation,
$$\frac{x - e^2(x+1)}{x+1} = 0 \iff \frac{(1-e^2)x - e^2}{x+1} = 0$$
4. Cancel the denominator and solve the numerator only:
$$x = \frac{e^2}{1-e^2} \approx -1.16.$$
5. Check the EC: OK.

Exercises

1. $\log_2\left(\frac{2}{x}\right) = 3$
2. $\log_{10}(x) = -1$
3. $\log_x(16) = 2$
4. $\log_2(x) + \log_2(x-2) = 3$
5. $\log_3(x) = 4$
6. $\ln\left(\frac{x+1}{x-1}\right) = 1$
7. $\log_{10}(100x) = 3$
8. $\log_2(x) + \log_2(x-3) = 2$

Solving Exponential equations using Logarithms I

Recall that $\log_a(a^x) = x$

Solve $3^x = 10$.

1. Apply the logarithm with base as that of the exponential (3) to both sides: $\log_3(3^x) = \log_3(10)$.
2. Use the power rule of logarithms: $x \log_3(3) = \log_3(10)$.

$$x \cdot 1 = \log_3(10)$$

Alternative way

You could solve it by applying the logarithm with base 10 or e (since you can directly find it in the calculator):

$$\ln(3^x) = \ln(10)$$

$$x \ln(3) = \ln(10) \iff x = \frac{\ln(10)}{\ln(3)} \approx 2.10$$

Solving Exponential equations using Logarithms II

Solve $e^x = 20$**.**

1. Apply the natural logarithm to both sides: $\ln(e^x) = \ln(20)$.
2. Use the power rule of logarithms: $x \ln(e) = \ln(20)$.

$$x = \ln(20)$$

Exercises

1. $4^x = 64$
2. $7^x = 50$
3. $10^{2x} = 1000$
4. $e^{x+2} = 50$
5. $e^{x^2} = 7$
6. $e^{2x^2} = 16$
7. $e^{x^2-1} = 5$
8. $e^{x^2+2x} = 10$

Exponential Inequalities

$$a^x > a^n$$

- if $a > 1$, the solution is $x > n$
- if $a < 1$, the solution is $x < n$

Example 1: $2^x > 8$

$$2^x > 2^3$$

$$x > 3$$

Example 2: $\left(\frac{1}{2}\right)^{2x} > 2^{-1}$

$$\left(\frac{1}{2}\right)^{2x} > \frac{1}{2}$$

$$2x < 1$$

$$x < \frac{1}{2}$$

Solving Exponential inequalities using Logarithms I

$$a^x > b = \begin{cases} \log_a(a^x) > \log_a(b) & \text{if } a > 1 \\ \log_a(a^x) < \log_a(b) & \text{if } a < 1 \end{cases}$$

Example 1: Solve $3^x \leq 15$.

1. Apply the logarithm to both sides: $\log_3(3^x) \leq \log_3(15)$.
2. Use the power rule of logarithms: $x \log_3(3) \leq \log_3(15)$.

$$x \leq \log_3(15)$$

Example 2: Solve $e^x > 7$.

1. Apply the natural logarithm to both sides: $\ln(e^x) > \ln(7)$.
2. Use the property $\ln(e^x) = x \iff x > \ln(7)$.

Solving Exponential inequalities using Logarithms II

Example 3: Solve $2^{2x+1} \geq 3$.

1. Apply the logarithm to both sides: $\log_2(2^{2x+1}) \geq \log_2(3)$.
2. Use the power rule: $(2x+1)\log_2(2) \geq \log_2(3)$.

$$2x + 1 \geq \log_2(3) \iff x \geq \frac{\log_2(3) - 1}{2}.$$

Solving Exponential inequalities using Logarithms III

Example 4: Solve $\left(\frac{1}{2}\right)^x \leq 5$.

1. Apply the logarithm to both sides. **The base of the log is less that 1:** we have to change the direction of the inequality

$$\log_{\frac{1}{2}}\left(\frac{1}{2}\right)^x \geq \log_{\frac{1}{2}}(5)$$

2. Use the power rule and solve it:

$$x\log_{\frac{1}{2}}\left(\frac{1}{2}\right) \geq \log_{\frac{1}{2}}(5)$$

$$x \geq \log_{\frac{1}{2}}(5)$$

Solving Exponential inequalities using Logarithms IV

Example 5: Solve $\left(\frac{1}{3}\right)^{2x+1} > 4$.

1. Apply the logarithm to both sides. **The base of the log is less than 1:** we have to change the direction of the inequality

$$\log_{\frac{1}{3}}\left(\frac{1}{3}\right)^{2x+1} < \log_{\frac{1}{3}}(4)$$

2. Use the power rule and solve it:

$$(2x+1)\log_{\frac{1}{3}}\left(\frac{1}{2}\right) < \log_{\frac{1}{3}}(4)$$

$$(2x+1) = \log_{\frac{1}{3}}(4) \iff x = \frac{\log_{\frac{1}{3}}(4) - 1}{2}.$$

Exercises

1. $\left(\frac{1}{8}\right)^x > \frac{1}{4}$
2. $250 \cdot 5^{\frac{x}{3}} > 2$
3. $\left(\frac{1}{5}\right)^{2x+1} < 625$
4. $\left(\frac{2}{5}\right)^{x+3} < \left(\frac{5}{2}\right)^{x-2}$
5. $e^{2x} - e^x > 6$
6. $e^{x+1} \leq 3e^x$
7. $e^{2x} + 2e^x - 3 > 0$

Logarithmic Inequalities I

$$\log_a(x) > b$$

with $a > 0$ and $x > 0$.
Immediate Solution Method: Rewrite in exponential form

$$\begin{cases} x > a^b & \text{if } a > 1 \\ x < a^b & \text{if } a < 1. \end{cases}$$

and solve it.
Example 1: Solve $\log_3(x) > 2$

- EC: $x > 0$
- Rewrite in exponential form: $x > 3^2$.
- **Solution:** $x > 9$. EC: OK

Logarithmic Inequalities II

Example 2: Solve $\log_{\frac{1}{2}}(x) > 3$

- EC: $x > 0$
- Rewrite in exponential form: $x < \left(\frac{1}{2}\right)^3 = \frac{1}{8}$.
- Intersect the solution with the EC. **Solution:** $x \in (0, \frac{1}{8})$.

Solving Logarithmic inequalities using Exponential I

Note

If we apply an exponential with base less than 1, we have to change the direction of the inequality.

Example 1: Solve $\log_5(x-1) \leq 1$

1. EC: $x > 1$
2. Apply the exponential to both sides: $5^{\log_5(x-1)} \leq 5^1$.
3. Simplify: $x - 1 \leq 5$
4. Solution of the equation: $x \leq 6$.
5. Intersect the solution with the EC: **Solution** $x \in (1, 6)$.

Solving Logarithmic inequalities using Exponential II

Example 2: Solve $-2 < \log_{\frac{1}{3}}(x)$

1. EC: $x > 0$.
2. Rearrange: $\log_{\frac{1}{3}}(x) > -2$
3. Apply the exponential to both sides. **The base of the exponential is less that** 1**:** we have to change the direction of the inequality

$$\left(\frac{1}{3}\right)^{\log_{\frac{1}{3}}(x)} > \left(\frac{1}{3}\right)^{-2}$$

$$x > 3^2 = 9$$

4. Check the EC: OK

Exercises

1. $\log_2(x^2 - 3x + 2) \geq 1$
2. $\ln(4x - 7) \leq 1$
3. $\log_{1/3}(x^2 - 5x + 6) < 2$
4. $\log_7(2x + 1) > -1$