Lecture 7
Absolute value and irrational Equations/Inequalities

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Lecture 7

1. Absolute value

2. Irrational equations

3. Irrational inequalities

Absolute value - Modulus

The **absolute value** or **modulus** of a number $a \in \mathbb{R}$ is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}.$$

It means that $|5| = 5$ and $|-5| = 5$.
Analogously, for every polynomial $A(x)$, it holds that

$$|A(x)| = \begin{cases} A(x), & \text{if } A(x) \geq 0 \\ -A(x), & \text{if } A(x) < 0 \end{cases}.$$

Example

$$|x-1| = \begin{cases} x-1, & \text{if } x-1 \geq 0 \\ -(x-1), & \text{if } x-1 < 0 \end{cases} = \begin{cases} x-1, & \text{if } x \geq 1 \\ -x+1, & \text{if } x < 1 \end{cases}$$

Equation with absolute value

Given an equation where the unknown is inside the modulus, the way to solve it is to split it in two equations where each of them exists in a different set of values of x.

$$|A(x)| = B(x) \tag{1}$$

The solution of (1) is the **union** of the solutions of the following two systems:

$$\begin{cases} A(x) \geq 0 \\ A(x) = B(x) \end{cases} \quad \bigcup \quad \begin{cases} A(x) < 0 \\ -A(x) = B(x) \end{cases}$$

Example

Solve $|2x-1| = x+7$

1. Write explicitly the modulus:

$$|2x-1| = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ -2x+1 & \text{if } x < \frac{1}{2} \end{cases}$$

2. *S1:*

$$\begin{cases} x \geq \frac{1}{2} \\ 2x-1 = x+7 \end{cases} = \begin{cases} x \geq \frac{1}{2} \\ x = 8 \end{cases}$$

   The solution of *S1* is $x = 8$.

3. *S2:*

$$\begin{cases} x < \frac{1}{2} \\ -2x+1 = x+7 \end{cases} = \begin{cases} x < \frac{1}{2} \\ x = -2 \end{cases}$$

   The solution of *S2* is $x = -2$.

The overall solutions are $x = 8$ and $x = -2$.

Inequality with absolute value

As it works for the equality with absolute value, in case of

$$|A(x)| > B(x) \tag{2}$$

the solution of (2) is equivalent to the union of the solutions of the following two systems

$$\begin{cases} A(x) \geq 0 \\ A(x) > B(x) \end{cases} \quad \bigcup \quad \begin{cases} A(x) < 0 \\ -A(x) > B(x) \end{cases}$$

In case of $<$ in (2), the direction of the second equation of each system change in the same way.

Example I

Solve $|x-4| > x^2$

1. Write explicitly the modulus:

$$|x-4| = \begin{cases} x-4 & \text{if } x \geq 4 \\ -x+4 & \text{if } x < 4 \end{cases}$$

2. *S1:*

$$\begin{cases} x \geq 4 \\ x-4 > x^2 \end{cases} = \begin{cases} x \geq 4 \\ x^2 - x + 4 < 0 \end{cases}$$

   The quadratic inequality has $\Delta < 0$ and it is concave up. It follows that it is always positive. There is not any solution.

Example II

3. *S2:*

$$\begin{cases} x < 4 \\ -x + 4 > x^2 \end{cases} = \begin{cases} x < 4 \\ x^2 + x - 4 < 0 \end{cases}$$

$$\begin{cases} x < 4 \\ -1.56 \approx \frac{1}{2}(-1 - \sqrt{17}) < x < \frac{1}{2}(\sqrt{17} - 1) \approx 1.56 \end{cases}$$

The solution of *S2* is $\frac{1}{2}(-1 - \sqrt{17}) < x < \frac{1}{2}(\sqrt{17} - 1)$

The overall solution is the solution of *(S2)*.

Exercises

1. $|x - 3| = 5$
2. $|2x + 4| = 8$
3. $|3x - 1| = 2x + 5$
4. $|x^2 - 4| = 0$
5. $|x - 2| \leq 3$
6. $|2x + 1| > 5$
7. $|3x - 4| \geq 2x + 1$
8. $|x + 3| < 2x - 1$
9. $|x^2 - 1| \leq x + 2$

Irrational equations

An **irrational equation** is an equation in which the unknown variable appears under a radical sign or with a fractional power

$$\sqrt[n]{A(x)} = B(x)$$

$$(A(x))^{\frac{1}{n}} = B(x)$$

where $A(x)$ and $B(x)$ are polynomial, and $n > 1$.

Note

We know that the **even root** of a (non-negative) number is non-negative. It follows that, if on the right-side we have a polynomial, we have to ask that $B(x) \geq 0$.

Resolution

1. *n* **odd:** isolate the radical (or the power) expression and raise both sides to the power n
$$\sqrt[n]{A(x)} = B(x)$$
$$A(x) = (B(x))^n$$
2. *n* **even:** we have to impose two existence conditions:
   - the polynomial above the root has to be non-negative: $A(x) \geq 0$
   - the result of an even root has to be non-negative: $B(x) \geq 0$

   Then, raise both sides to the power n to eliminate the radical or the power and solve the resulting equation. The result has to satisfy the existence conditions.

Example 1

Solve the equation: $\sqrt{x+1} = x - 1$

1. Write the EC:
   - $(x+1) \geq 0$, i.e., $x \geq -1$
   - $(x-1) \geq 0$, i.e., $x \geq 1$.

   Intersection of EC: $x \geq 1$**.**
2. Square both sides:
$$(\sqrt{x+1})^2 = (x-1)^2$$
3. Simplify: $x + 1 = x^2 + -2x + 1$
4. Rearrange: $x^2 - 3x = 0 \iff x(x-3) = 0$
5. The solutions are
$$x_1 = 0, \quad x_2 = 3$$
6. Check the EC: NO. $x_1 = 0$ is **not** a solution.
7. **Solution:** $x = 3$.

Example 2

Solve the equation: $\sqrt{2x+3} = -1$

1. Write the EC:
   - $(2x+3) \geq 0$, i.e., $x \geq -\frac{3}{2}$
   - $-1 \geq 0$, **is impossible**

   It follows that the equation **is impossible**, since for every $x \geq -\frac{3}{2}$, the square root gives a result that is greater or equal to 0. Thus, it cannot be -1.

Example 3

Solve the equation:

$$(2x+3)^{\frac{1}{3}} = 2$$

1. n is odd, thus we don't have to write the EC
2. Raise both sides to the power $\frac{1}{n} = 3$,

$$\left((2x+3)^{\frac{1}{3}}\right)^3 = (2)^3$$

3. Simplify:

$$2x+3=8$$

4. Rearrange and solve:

$$2x = 5, \quad x = \frac{5}{2}$$

Exercises

1. $\sqrt{x^2 - 4x + 3} = 2x - 2$
2. $\sqrt[3]{4x^2 + 8x - 2x} = 2x - 1$
3. $2\sqrt{2 + x} - \sqrt{x - 3} = 4$

Irrational inequalities

$$\sqrt[n]{A(x)} > B(x) \Leftrightarrow (A(x))^{\frac{1}{n}} > B(x)$$

where $A(x)$ and $B(x)$ are polynomial, and $n > 1$.

Procedure:

(1) n **odd:** no EC. Raise at power n both sides and solve the inequality

(2) n **even:**,

(2.1) $\sqrt[n]{A(x)} > B(x)$: the solution is the union of the solutions of the two systems

$$\begin{cases} B(x) > 0 \\ A(x) > (B(x))^n \end{cases} \bigcup \begin{cases} B(x) < 0 \\ A(x) > 0 \end{cases}$$

(2.2) $\sqrt[n]{A(x)} < B(x)$: the solution is the solutions of the system

$$\begin{cases} B(x) \geq 0 \\ A(x) < (B(x))^n \\ A(x) > 0 \end{cases}$$

Recall: Systems of Inequalities

Given a set of two or more inequalities with the same variable x, we look for the solution of each inequality and then the overall solution is the **region where the solutions of all inequalities overlap**.

Example

$$\begin{cases} 2x+4 & > 0 \\ x-3 & \geq 0 \\ 5-x & > 0 \end{cases}$$

1. Solve each inequality separately:
   - $x > -2$
   - $x \geq 3$
   - $x < 5$
2. Overlap the solutions graphically:

Solution: $x \in [-3, 5)$.

Example

Solve the inequality:

$$\sqrt{3x+9} < x-3$$

1. **Note: it is not necessary write the EC since they are included in the resolution procedures**
2. Case *(2.2)*

$$\begin{cases} x-3 \geq 0 \\ 3x+9 < (x-3)^2 \\ 3x+9 > 0 \end{cases} \qquad \begin{cases} x \geq 3 \\ x^2-6x-3x > 0 \\ x > -3 \end{cases} \qquad \begin{cases} x \geq 3 \\ x < 0 \text{ and } x > 9 \\ x > -3 \end{cases}$$

Solution:

$$x \in (9, +\infty).$$

Example

Solve the inequality: $\sqrt{34 - 3x} > x - 2$
Case 2.1

$$\begin{cases} x - 2 < 0 \\ 34 - 3x > 0 \end{cases} \bigcup \begin{cases} x - 2 \geq 0 \\ 34 - 3x > (x-2)^2 \end{cases}$$

$$\begin{cases} x < 2 \\ x < \frac{34}{3} \end{cases} \bigcup \begin{cases} x \geq 2 \\ -5 < x < 6 \end{cases}$$

$$x \in (-\infty, 2) \cup [2, 6) \Rightarrow x \in (-\infty, 6)$$

Exercises

1. $\sqrt[3]{x} < x$
2. $\sqrt{x-3} < 2x - 1$
3. $\sqrt{2x+15} \leq x$
4. $\sqrt{x^2 - 7x} < x - 2$
5. $\sqrt{4x^2 - 4x - 8} > x + 1$