Lecture 5-6
Rational Equations and Inequalities

Preliminary course - Mathematics 2026/2027
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Equations

An equation is an equality between two polynomials where there is one or more unknowns for which we look for the values that make the equality true, that are named **solutions** or **roots**.

Consider the equation:

$$3x + 10 = x + 4$$

We want to find all values of x that satisfy the equality. They will be called **solutions** or **roots**.

- **attempt 1:** if we choose $x = 5$,

$$3x + 10 = 25, \quad \text{but} \quad x + 4 = 9.$$

  We have $25 \neq 9$, thus $x = 5$ is not a solution
- **attempt 2:** if we choose $x = -3$,

$$3x + 10 = 1 \quad \text{and} \quad x + 4 = 1.$$

  We have $1 = 1$, thus $x = -3$ is a solution (root).

Solving equations I

Definition: equivalence

Two or more equations are **equivalent** if they have the same solutions

Two operations come in handy when we try to solve equations:

I equivalence principle: Adding or subtracting the same term from both sides the equation is equivalent.

$$A = B \Longleftrightarrow A + C = B + C$$

It allows us to use the following techniques:

$\Longrightarrow$ We can move a term from one side to the other by changing its sign

$$A + B = C \Longleftrightarrow A = C - B$$

$\Longrightarrow$ We can remove identical terms that are on both sides

$$A + \cancel{B} = C + \cancel{B} \Longleftrightarrow A + C$$

Solving equations II

II equivalence principle: Multiplying or dividing both sides by a non-zero term the equation is equivalent.

$$A = B \Longleftrightarrow A \cdot C = B \cdot C, \quad C \neq 0$$

It allows us to use the following techniques:

$\Longrightarrow$ We can change all the term's signs:

$$A = B \Longleftrightarrow -A = -B$$

Example

Solve: $3x + 10 = x + 4$

1. subtract 10 from both sides

$$3x + 10 - 10 = x + 4 - 10$$

$$3x = x - 6$$

2. subtract x from both sides

$$3x - x = x - 6 - x$$

$$2x = -6$$

3. divide both sides by 2

$$2x\frac{1}{2} = -6\frac{1}{2}$$

$$x = \frac{-6}{2} = -3.$$

Equations/Inequalities:

- algebraic
  - rational: polynomial $A(x)$ or fractional $\frac{A(x)}{B(x)}$
  - irrational: the unknown appears under the sign of a root or is raised to power with a fractional exponent
- trascendental
  - log
  - exp
  - trigonometric

The degree of an equation/inequality is the degree of the polynomial that composes it.

Rational eq.: 1^{st} degree equations (linear equations) I

$$ax + b = 0$$

- a is the coefficient
- x is the unknown (or variable)
- b is the known term (constant term)

Rational eq.: 1^{st} degree equations (linear equations) II

Example:

$$\frac{1}{3}x - (2x+1) = 3(2-x) - \frac{1}{2}$$
$$\frac{1}{3}x - 2x - 1 = 6 - 3x - \frac{1}{2}$$
$$\frac{1}{3} - 2x + 3x = 6 - \frac{1}{2} + 1$$
$$(\frac{1}{3} - 2 + 3)x = 7 - \frac{1}{2}$$
$$(\frac{1}{3} + 1)x = \frac{14-1}{2}$$
$$\frac{1+3}{3}x = \frac{13}{2}$$
$$\frac{4}{3}x = \frac{13}{2}$$
$$\frac{3}{4}\frac{4}{3}x = \frac{3}{4}\frac{13}{2}$$
$$x = \frac{39}{8}.$$

Particular cases

1. $0x = k$, with $k \neq 0$, is impossible. For every $x \in \mathbb{R}$, $0x = 0$, thus it cannot be $k\ ne0$.
2. $kx = 0$ has solution $x = 0$.
3. $0x = 0$ is true for every value you can assign to x. Thus the solution is **indeterminate**, that is, $\forall x \in \mathbb{R}$.

Exercises

1. $$7(2x-3)-4(3x+2)=5x-8$$
2. $$\frac{2x-5}{3}+\frac{x+7}{4}=3$$
3. $$3(2x-4)+5(x+3)-2(x-7)=4x+12-3(x-5)$$
4. $$x(x+7)+9=x+(x+3)^2$$

$$ax + b > 0$$

Rules of linear equations hold.

Note

Multiplying/dividing both sides for a negative number, the direction of the inequality changes!

- if $a > 0$,

$$x > -\frac{b}{a} \Leftrightarrow x \in (-\frac{b}{a}, +\infty)$$

- if $a < 0$,

$$x < -\frac{b}{a} \Leftrightarrow x \in (-\infty, -\frac{b}{a})$$

Example

Solve: $\frac{1}{3}x - (2x+1) > 3(2-x) - \frac{1}{2}$

$$\frac{1}{3}x - 2x - 1 > 6 - 3x - \frac{1}{2}$$

$$\frac{1}{3} - 2x + 3x > 6 - \frac{1}{2} + 1$$

$$(\frac{1}{3} - 2 + 3)x > 7 - \frac{1}{2}$$

$$(\frac{1}{3} + 1)x > \frac{14-1}{2}$$

$$\frac{1+3}{3}x > \frac{13}{2}$$

$$\frac{4}{3}x > \frac{13}{2}$$

$$\frac{3}{4}\frac{4}{3}x > \frac{3}{4}\frac{13}{2}$$

$$x > \frac{39}{8} \Leftrightarrow x \in \left(\frac{39}{8}, +\infty\right)$$

Example

Solve: $-6x + 9(2 - x) < -3x - 2$

$$-6x + 18 - 9x + 3x < -1$$

$$-12x < -2 - 18$$

$$\frac{1}{-12}(-12x) > \frac{1}{-12}(-20)$$

$$x > \frac{20}{12} = \frac{5}{3}$$

$$x \in \left(\frac{5}{3}, +\infty\right)$$

Rational eq.: 2^{nd} degree equations (quadratic equations) I

$$ax^2 + bx + c = 0$$

Cases:

incomplete: $c = 0$, the equation reduces to $ax^2 + bx = 0$. It can be factorized and we get

$$x(ax + b) = 0,$$

whose solutions are $x = 0$ and $(ax + b) = 0$, that is a 1^{st} degree equation.

Rational eq.: 2^{nd} degree equations (quadratic equations) II

pure $b = 0$, the equation reduces to $ax^2 + c = 0$. The solutions are (if $-c/a \geq 0$)

$$ax^2 = -c$$
$$x^2 = -\frac{c}{a}$$
$$x = \pm\sqrt{-\frac{c}{a}}.$$

1^{st} degree $a = 0$, it reduces to a 1^{st} degree equation

monomial $b = c = 0$. It reduces to $x^2 = 0$, whose root is $x = 0$

Rational eq.: 2^{nd} degree equations (quadratic equations) III

complete $a, b, c \neq 0$. The solutions are given by the **quadratic formula**

$$x_1, x_2 = \frac{-b \pm \sqrt{\Delta}}{2a}$$

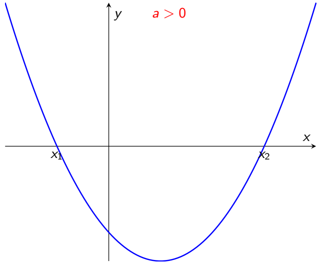
where $\Delta = b^2 - 4ac$.

The graph of a quadratic equation is a **parabola** that is **concave up** if $a > 0$, **concave down** if $a < 0$.

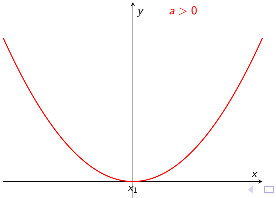
The discriminant Δ determines the nature of the roots.

Discriminant cases I

- $\Delta > 0$: two real distinct roots x_1 and x_2. The parabola intersects the x-axis at two points.

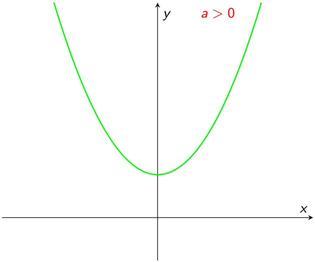


- $\Delta = 0$: two real roots that coincide $x_1 = x_2$. The parabola is tangent to the x-axis at one point.



Discriminant cases II

- $\Delta < 0$: the equation has no real roots. It means that the parabola does not intersect the x-axis.



Zero-product property

If we have (or we get through factorization) an equation like

$$A(x) \cdot B(x) = 0,$$

it is true if and only if

$$A(x) = 0, \quad \text{or} \quad B(x) = 0.$$

It follows that the solutions are the union of the roots.
Example:

$$(x+3)(x-1) = 0$$

- $x + 3 = 0$, that is $x = -3$
- $x - 1 = 0$, that is $x = 1$

The **solutions** are $x = 1$ and $x = -3$.

Exercises

1. $x^2 - 3x + 2 = 0$
2. $x^2 + x + 1 = 0$
3. $2x^2 + 4 = 0$
4. $x^2 - 4x + 4 = 0$
5. $(3x-1)^2 + (4x+1)^2 = (5x-1)(5x+1) + 1$

2^{nd} degree inequalities (quadratic inequalities)

A quadratic inequality has the form:

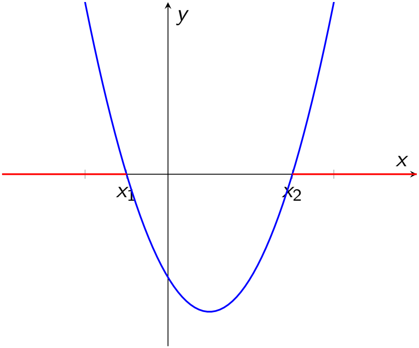
$$ax^2 + bx + c > 0$$

Procedure: Compute the discriminant of the associate equation. After have found the solution of the associate equation, x_1 and x_2, find the solution of the inequality by studying the sign of the discriminant and the concavity of the parabola.

Case: $\Delta > 0$

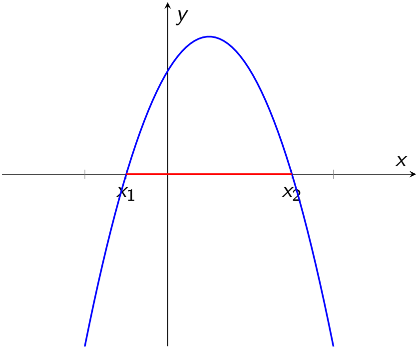
$$ax^2 + bx + c > 0$$

Concave Up ($a > 0$)



Solution: $x \in (-\infty, x_1) \cup (x_2, +\infty)$

Concave Down ($a < 0$)

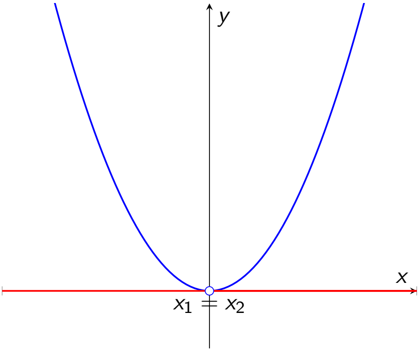


Solution: $x \in (x_1, x_2)$

Case: $\Delta = 0$

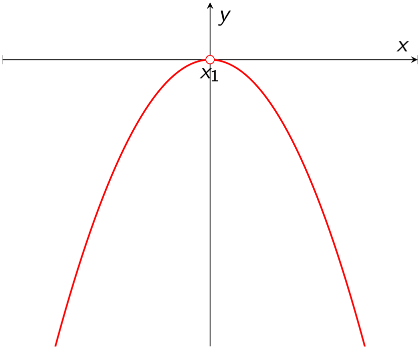
$$ax^2 + bx + c > 0$$

Concave Up ($a > 0$)



Solution: $x \in \mathbb{R} \setminus \{x_1\}$,
$x \in (-\infty, x_1) \cup (x_1 + \infty)$

Concave Down ($a < 0$)

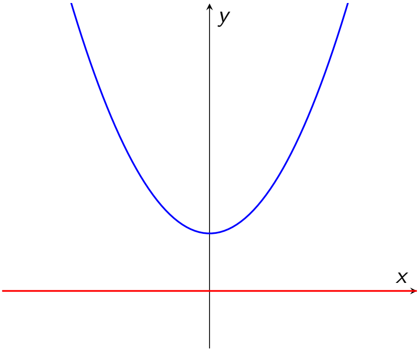


Solution: $\nexists x$

Case: $\Delta < 0$

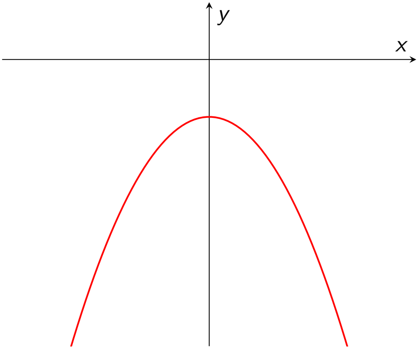
$$ax^2 + bx + c > 0$$

Concave Up ($a > 0$)



Solution: $\forall x \in \mathbb{R}$

Concave Down ($a < 0$)



Solution: $\nexists x$

Examples

Solve the inequality $6x^2 - 5x + 1 \geq 0$

1. $a = 6 > 0$, thus the parabola is concave up
2. Compute the discriminant of the associate equation: $6x^2 - 5x + 1 = 0$. We get $\Delta = 1$
3. It means that the parabola is concave up and it intersects the x-axis at two points, that are $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{2}$.
4. Case $\Delta > 0$, $a > 0$.

Solution $x \in \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{2}, +\infty\right)$.

Solve the inequality $-3x^2 + 5x - 2 \geq 0$

1. $a = -3 < 0$, thus the parabola is concave down
2. Compute the discriminant of the associate equation: $-3x^2 + 5x - 2 = 0$. We get $\Delta = 1$
3. It means that the parabola is concave down and it intersects the x-axis at two points, that are $x_1 = \frac{2}{3}$, $x_2 = 1$.
4. Case $\Delta > 0$, $a < 0$. **ATTENTION: we are looking for the** $\geq$.

Solution $x \in \left[\frac{2}{3}, 1\right]$.

Exercises

1. $$x^2 - 4x - 5 < 0$$

2. $$2x^2 + 3x - 2 \geq 0$$

3. $$x^2 - 6x + 9 > 0$$

4. $$x^2 + 4x + 5 < 0$$

Rational equations with degree higher than the second I

We don't have a closed form expression for the roots

We have to factorize the equation using the factorization technique and use the **Zero-product property**: the roots of an equation in the form

$$A(x) \cdot B(x) \cdot \ldots \cdot P(x) = 0$$

are the **union** of the roots of

- $A(x) = 0$,
- $B(x) = 0$,
- ...
- $P(x) = 0$.

Rational equations with degree higher than the second II

Examples:

1. $x^3 - 2x = 0$. It can be factorized as $x(x^2 - 2) = 0$.
   Roots: $x_1 = 0$ and $x^2 = 2$, that is $x_2 = +\sqrt{2}$, $x_3 = -\sqrt{2}$.
2. $x^4 - 6x^2 + 9 = 0$. Recognize the square of a binomial, that is $(x^2 - 3)^2 = 0$. It follows that $(x^2 - 3) = 0$, whose roots are $x_1, x_2 = \pm\sqrt{3}$.
3. $x^3 - 6x^2 + 11x - 6 = 0$. With the factorization through the division by columns (Ruffini's rule), we get $(x^2 - 5x + 6)(x - 1)$ (**Check the factorization!**).
   - $x^2 - 5x + 6 = 0$, through the quadratic formula we get $x_1 = 2$ and $x_2 = 3$
   - $(x - 1) = 0$, that is $x_3 = 1$.

Fractional equations

The unknown is at the denominator, that is

$$\frac{A(x)}{B(x)} = 0,$$

where $A(x)$ and $B(x)$ are polynomial.

In this case we have to write the **existence conditions** (EC): conditions on the unknown under which the equation has valid solution, that is $B(x) \neq 0$.

For example: given the equation $\frac{2}{x-1} + 3 = 0$.
If we take $x = 1$, we get $\frac{2}{1-1} + 3 = 0$, that is $\frac{2}{0} + 3$. The division $2 \div 0$ is undefined. It follows that $x = 1$ cannot be a solution of the equation.

Examples

Solve the equation $\frac{2x}{x+2} + x - 3 = 0$.

1. Write the existence conditions: $x + 2 \neq 0$, that is $x \neq -2$
2. Write the equation above the same denominator, through the LCD
$$\frac{2x + x(x+2) - 3(x+2)}{x+2} = \frac{2x + x^2 + 2x - 3x - 6}{x+2} = 0$$
3. Once written the existence conditions, we can multiply both sides for the denominator (since the EC assure that it is not 0). In this way we **always** cancel the denominator
$$(x+2)\frac{2x + x^2 + 2x - 3x - 6}{x+2} = (x+2)0$$
$$2x + x^2 + 2x - 3x - 6 = 0$$
4. Solve the equation at the numerator
$$x^2 + x - 6 = 0$$
$$\Delta = (1)^2 - 4(1)(-6) = 25$$
$$x_1, x_2 = \frac{-1 \pm \sqrt{25}}{2}$$
$$x_1 = 2, \quad x_2 = -3.$$
5. Check if the solutions satisfies the EC. In this case, YES

Examples

Solve the equation $\frac{3x+1}{x-2} = \frac{x+6}{3x-6}$.

1. EC: $x - 2 \neq 0$, that is $x \neq 2$; $3x - 6 \neq 0$, that is $x \neq 2$.
2. Let's take everything to one side and factorize the denominator (if possible):
$$\frac{3x+1}{x-2} - \frac{x+6}{3(x-2)} = 0$$
3. Write the equation above the same denominator. Factorize the denominators and write the LCD. In this case
$$LCD(x-2, 3(x-2)) = 3(x-2).$$
$$\frac{3(3x+1) - (x+6)}{3(x-2)} = 0$$
4. Cancel the denominator and solve the numerator
$$9x + 3 - x - 6 = 0$$
$$8x - 3 = 0$$
$$x = \frac{3}{8}.$$
5. Check if it satisfies the EC: YES

Exercises

1. $$\frac{2x+1}{x-3} = 1$$

2. $$\frac{x^2-4}{x+2} = x-2$$

3. $$\frac{x^3-3x}{x-1} = x^2+2x$$

4. $$\frac{2x^2-x-1}{x^2-1} = \frac{x+1}{x+1}$$

5. $$\frac{x-2}{x^2-x} - \frac{1}{x^2-3x+2} = \frac{x-1}{x(x-2)}$$

Fractional Inequalities

$$\frac{A(x)}{B(x)} > 0$$

where $A(x)$ and $B(x)$ are polynomial.

As in the case of equations, we have to write the **existence conditions**, i.e., when $B(x) \neq 0$.

Procedure:

1. Solve separately the inequality at the numerator and at the denominator (**ATTENTION: contrary to the equation case, here we cannot delete the denominator!**)
2. Combine solution's intervals of the numerator and of the denominator to find the entire solution set, with the use of the **sign diagram**.

Example

Solve the inequality: $\frac{x+1}{x-2} > 1$

1. EC: $x - 2 \neq 0$, i.e., $x \neq 2$
2. $\frac{x+1}{x-2} - 1 > 0$
$$\frac{3}{x-2} > 0$$
3. Study the sign of the numerator: $3 > 0$ is always true, for every $x \in \mathbb{R}$
4. Study the sign of the denominator: $x - 2 > 0$, i.e., $x > 2$
5. Combine the solutions in the sign diagram

	$x < 2$	$x > 2$
Numerator	$+$	$+$
Denominator	$-$	$+$
Overall Sign	$-$	$+$

6. The equation is positive for $x > 2$. Check if it is coherent with the CE: YES

Solution: $x \in (2, +\infty)$.

Example 2

Solve the inequality: $\frac{2x}{x^2-1} \leq 0$

1. EC: $x^2 - 1 \neq 0$, i.e., $x \neq \pm 1$.
2. Study the sign of the numerator (**ATT:** always study the sign $>$): $2x > 0$, i.e., $x > 0$
3. Study the sign of the denominator (**ATT: sign** $>$): $x^2 - 1 > 0$. It is a parabola concave up with $\Delta > 0$. The solutions are $x_1 = -1, x_2 = 1$. The denominator is positive for $x < -1$ and $x > 1$. is
4. Combine the solutions in the sign diagram

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$x > 1$
N: $x > 0$	$-$	$-$	$+$	$+$
D: $x < -1$ and $x > 1$	$+$	$-$	$-$	$+$
Overall Sign	$-$	$+$	$-$	$+$

5. The equation would be less than or equal to 0 for $x \leq -1$ and $0 \leq x \leq 1$.
6. Check if it satisfies the EC: NO. $x \neq \pm 1$.

Solution: $x \in (-\infty, -1) \cup [0, 1)$

Exercises

1. $$\frac{x-3}{x^2+x-6} \geq 0$$
2. $$\frac{2x+1}{x^2-4x+3} \leq 0$$
3. $$\frac{3x-4}{x^2-x-6} > 0$$
4. $$\frac{x^2-1}{2x-3} \leq 0$$

Rational inequalities with degree higher than the second I

As for the equation case, we don't have a closed form expression for the roots.
We have to factorize the associate equation using the factorization techniques.
Given the following form

$$(\text{expr } 1) \cdot (\text{expr } 2) \cdot \cdots \cdot (\text{expr } n) > 0$$

the solution is given by the combination of the solution's interval of each expression through the **sign diagram**.

Example

Solve $x^3 - 2x \leq 0$.

1. It can be factorized as $x(x^2 - 2) \leq 0$
2. Study the sign of each expression separately (with $>$ sign)
   - $x > 0$
   - $x^2 - 2 > 0$. It is a concave up parabola with roots $x_1 = -\sqrt{2}$, $x_2 = \sqrt{2}$. Its solution is $x < -\sqrt{2}$ or $x > \sqrt{2}$.
3. Combine the solutions:

	$x < -\sqrt{2}$	$-\sqrt{2} < x < 0$	$0 < x < \sqrt{2}$	$x > \sqrt{2}$
$(x > 0)$	$-$	$-$	$+$	$+$
$(x^2 - 2 > 0)$	$+$	$-$	$-$	$+$
Overall Sign	$-$	$+$	$-$	$+$

Solution: $x \in (-\infty, -\sqrt{2}] \cup [0, \sqrt{2}]$.

Exercises

1. $x^4 - 81 < 0$
2. $x^3 - 4x^2 - x + 4 \geq 0$
3. $x^4 - 5x^2 + 4 \leq 0$