# Lectures 3 and 4
# Monomials and polynomials

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# Lectures 3-4

# Monomials

A monomial is an expression containing **only** a numerical part and a literal part, multiply among themselves,

$$2 \cdot a, \quad -3 \cdot a^4 \cdot b, \quad \cancel{3 \cdot a + b}, \quad \cancel{\frac{1}{b}}.$$

- the number is called **coefficient**
- the letters are called **variables**
- the **degree** of the monomial with respect to a variable is the exponent of the variable
- the **total degree** of the monomial is the sum of the exponents of all the variables

**NO SUM, NO LITERAL PART IN THE DENOMINATOR(i.e., not fractional exponents)**

# Operations I

## Algebraic sum:

This operation can be done only between monomials that are **similar**, that is, they have the same literal part (same variables and same exponents).
**Procedure:** compute the algebraic sum of the coefficients and maintain the literal part as it is.

1. $5ab$ and $3a$ are **not** similar
2. $-3a^3b$ and $4a^3b$ are similar. **Operations:**
   - $-3a^3b + 4a^3b = (-3+4)a^3b = 1a^3b$
   - $-3a^3b - 4a^3b = (-3-4)a^3b = -7a^3b$

# Operations II

## Multiplication

This operation can always be done.
**Procedure:** the product of two monomials is the monomial that has as its coefficient the product of the coefficients and as its literal part the product of the literal parts.

Rules and properties of powers, fractions, roots, continue to hold for the literal part.

1. $$(5ab)\cdot(3a) = (5\cdot 3)(a\cdot a)b = 15a^2b$$

2. $$\left(-\frac{1}{2}x^2y\right)^3\cdot(2yz^2) = \left(-\frac{1}{2^3}x^6y^3\right)\cdot(2yz^2) = \left(-\frac{1}{2^3}\cdot 2\right)(x^6)(y^3\cdot y)(z^2)$$
$$= -\frac{1}{2^2}x^6y^4z^2$$

# Operations III

## Division:

This operation can be done **only if** the divisor has neither letters that are not present in the dividend, nor letters that have exponents greater than those of the dividend.

**Procedure:** the ratio of two monomials is the monomial that has as its coefficient the ratio of the coefficients and as its literal part the ratio of the literal parts.

1. $$(15x^2y) : (3x) = \frac{15}{3}\frac{x^2}{x}\frac{y}{1} = 5xy$$

2. $$(15x^2) : (3xy) = (15x^2) \cdot \frac{1}{3xy} = \frac{15x^2}{3xy} = \frac{5x}{y}.$$

It is **correct** as an operation, but it is **not a monomial**.

# LCM between monomials

Just as with numbers, the least common multiple (LCM) of two monomials is the smallest monomial that is divisible by both of the original monomials

**Procedure:**

1. find the LCM of the coefficients
2. for each variable that **can be both common or uncommon in either monomials**, take the highest exponent
3. multiply the LCM of the coefficients by by each variable raised to its maximum exponent.

**Example:**

$$LCM(6a^2b^3, 9ab^2c) = 18a^2b^3c$$

# GCD between monomials

The Greatest Common Divisor (GCD) of two monomials is the largest monomial that divides both original monomials exactly (i.e., without remainder).

**Procedure:**

1. find the GCD of the coefficients
2. for each variable that appears in **both monomials**, take the lowest exponent
3. multiply the GCD of the coefficients by each common variable raised to its minimum exponent

**Example:**

$$GCD(6a^2b^3, 9ab^2c) = 3ab^2.$$

A polynomial is the algebraic sum of monomials

$$a+b-c, \quad -2x+y, \quad \cancel{3a^{-1}}, \quad \cancel{\frac{2}{x}}.$$

- if it is composed by two [three] monomials it is called **binomial [trinomial]**
- the **degree** of a polynomial is the highest degree of its monomial; ex.$3x^3y+y-\frac{1}{2}xy^5$ has degree 6.
- if all its monomials have the same degree, the polynomial is called **homogeneous**

**NO LITERAL PART IN THE DENOMINATOR**

# Algebraic sum

To add two (or more) polynomials, simply add **similar** monomials, i.e., terms with the same variables raised to the same powers.
**Example:**

$$(3x^2+2x+5)+(x^3+4x^2-3x+1) = (0+x^3)+(3x^2+4x^2)+(2x-3x)+(5+1)$$
$$= x^3 + 7x^2 - 1x + 6$$

# Multiplication

**Procedure:** Multiply all the terms of the first polynomial by all the terms of the second polynomial.

1.

$$9xy + 3x(1-3y) + 3x - x(2y)^2 = 9xy + 3x - 9xy + 3x - x(4y^2)$$

$$9xy + 3x - 9xy + 3x - 4xy^2$$

$$(9-9)xy + (3+3)x - 4xy^2 = 6x - 4xy^2$$

2.

$$(a+b)(a^2-b) - (a^2+1)(a-b) - (b-a)$$

$$a^3 - ab + a^2b - b^2 - (a^3 - a^2b + a - b) - b + a)$$

$$a^3 - ab + a^2b - b^2 - a^3 + a^2b - a + b - b + a$$

$$-ab + 2a^2b - b^2.$$

# Special products

difference of square $a^2 - b^2 = (a+b)(a-b)$

square of binomial $(a \pm b)^2 = a^2 + b^2 \pm 2ab$

cube of binomial $(a \pm b)^3 = a^3 \pm b^3 \mp 3a^2 b \pm 3ab^2$

square of trinomial $(a \pm b \pm c)^2 = a^2 + b^2 + c^2 \pm 2ab \pm 2ac \pm 2bc$

difference of cube $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

# Exercises: solve the following special products

1. $4a^2 - 1$
2. $(xy - 1)^2$
3. $16a^2x^6 - 9$
4. $25a^6b^8 - \frac{1}{4}$
5. $8a^3 + b^3$
6. $(\frac{1}{2}x - 3a)^3$
7. $x^3 - 64y^3$

# Division of a polynomial by a monomial I

Divide each term of the polynomial by the monomial

**Example:**

$$(6x^3 + 9x^2 - 3x) : (3x) = \frac{6x^3 + 9x^2 - 3x}{3x}$$

1. Divide each term:
$$\frac{6x^3}{3x}, \quad \frac{9x^2}{3x}, \quad \frac{-3x}{3x}$$
2. Simplify each term:
$$2x^2, \quad 3x, \quad -1$$
3. Write the simplified result:
$$2x^2 + 3x - 1$$

**NOTE: it means that we can write the dividend as**

$$6x^3 + 9x^2 - 3x = (2x^2 + 3x - 1)(3x),$$

**that is a decomposition!**

# Division of a polynomial by a monomial II

$$(10x^4 - 15x^3 + 5x^2) : (5x^2) = \frac{10x^4 - 15x^3 + 5x^2}{5x^2}$$

1. Divide each term:
$$\frac{10x^4}{5x^2}, \quad \frac{-15x^3}{5x^2}, \quad \frac{5x^2}{5x^2}$$
2. Simplify each term:
$$2x^2, \quad -3x, \quad 1$$
3. Write the simplified result:
$$2x^2 - 3x + 1$$

# Polynomial Division by column

It is analogous to the long division of numbers.

$$\frac{A(x)}{B(x)} = Q(x) + \frac{R(x)}{B(x)} \iff A(x) = B(x)Q(x) + R(x)$$

- $A(x)$ (dividend) and $B(x)$ (divisor) are two polynomial with $x$ as variable
- $Q(x)$ is the result of the division: **quotient**
- $R(x)$ is what is left over: **remainder**

# Example

Divide $(2x^3 + 4x - 5)$ by $(x - 1)$

$$
\begin{array}{cccc|l}
2x^3 & 0 & +4x & -5 & x-1 \\ \cline{5-5}
-2x^3 & +2x^2 & & & = 2x^2 + 2x + 6 \\ \cline{1-4}
0 & 2x^2 & +4x & -5 & \\
 & -2x^2 & +2x & & \\ \cline{2-4}
 & 0 & 6x & -5 & \\
 & & -6x & +6 & \\ \cline{3-4}
 & & & 1 &
\end{array}
$$

1. Divide the first term: $\frac{2x^3}{x} = 2x^2$
2. Multiply $(2x^2) \cdot (x - 1)$ and subtract:

$$(2x^3 + 4x - 5) - (2x^3 - 2x^2)$$

$$= 2x^2 + 4x - 5$$

3. Repeat with the new first term: $\frac{2x^2}{x} = 2x$
4. Multiply $(2x) \cdot (x - 1)$and subtract:

$$(2x^2 + 4x - 5) - (2x^2 - 2x) = 6x - 5$$

5. Repeat: $\frac{6x}{x} = 6$
6. Multiply $(6) \cdot (x - 1)$ and subtract:

$$(6x - 5) - (6x - 6) = 1$$

# Note

It holds that

$$\frac{2x^3+4x-5}{x-1} = 2x^2+2x+6+\frac{1}{x-1}$$

**NOTE: given the ratio of two polynomial, you can split the numerator but you cannot split the denominator**, that is

$$\frac{2x^3+4x-5}{x-1} = \frac{2x^3}{x-1}+\frac{4x}{x-1}+\frac{-5}{x-1}$$

$$\frac{2x^3+4x-5}{x-1} \neq \frac{2x^3+4x-5}{x}+\frac{2x^3-3x^2+4x-5}{-1}$$

# Another Example

**Divide** $(x^4 - 6x^3 + 11x^2 - 6x)$ **by** $(x^2 - 2)$

# Another Example

**Divide** $(x^3 + 2 - 3x)$ **by** $(x + 2)$

# Polynomial factorization

Our aim is the **decomposition** of a polynomial as a product of polynomial with lower degree.

**Methods:**

- total grouping
- partial grouping
- recognizing special products
- Polynomial division (Ruffini's theorem)

# Total grouping

$$3x^3 - 6x^2 + 9x = \text{ GCD } \cdot \text{ sum of quotients}$$

1. Find the GCD: $GCD(3x^3, 6x^2, 9x) = 3x$.
2. Divide each monomial by the GCD:

$$\frac{3x^3}{3x} = x^2, \quad \frac{-6x^2}{3x} = -2x \quad \frac{9x}{3x} = 3.$$

   They form another polynomial that is the quotient: $(x^2 - 2x + 3)$
3. (common factor) $\cdot$ (new polynomial)

$$3x(x^2 - 2x + 3).$$

# Partial grouping

If there is no GCD among all monomials but there is among groups of monomials, we can proceed as before (GCD $\cdot$ quotient) in groups.
**The aim is to get a new GCD that can allow a total grouping on the second step!**
**Example:**

$$4x^2 - 8bx + 3x - 6b$$

$$GCD(4x^2, 3x) = x, \qquad GCD(-8bx, -6b) = -2b$$

$$x(4x+3) - 2b(4x+3)$$

We get a new common term that is $(4x+3)$ and we can proceed with a new total grouping:

$$(4x+3)(x-2b).$$

# Recognize a special product

binomial

- $x^2 - y^2 = (x - y)(x + y)$
- $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
- $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

trinomial

- $x^2 + y^2 + 2xy = (x + y)^2$
- $x^2 + y^2 - 2xy = (x - y)^2$

quadrinomial

- $x^3 + y^3 + 3x^2y + 3xy^2 = (x + y)^3$
- $x^3 - y^3 + 3x^2y - 3xy^2 = (x - y)^3$

# Exercises: factorize the following expressions

1. $x^2(2-6x)-2y^2(4-12x)$
2. $2x^3-2x^2-12x+12$
3. $3x^3-12x^2+12x$
4. $-x^4-9+6x^2$
5. $5x^2-20+x^4-4x^2$
6. $5ab-10b^2+3a^2b-6ab^2$
7. $a^2bx+a^2b+bxy^2+by^2$
8. $y^4-y^3-2y+2$
9. $9y^2+\frac{1}{4}-3y$

# Factorization with polynomial division (Ruffini's theorem) I

We have an **ordered** polynomial of degree $n$ with $p$ coefficients.

$$A(x) = a_1 x_1^n + a_2 x_2^{n-1} + \ldots + a_p x_p^0.$$

## Aim

We want to divide $A(x)$ by a binomial of the form $(x-k)$ in order to get a quotient $Q(x)$ of lower degree and $R(x) = 0$. In this way we can factorize the polynomial $A(x)$ as

$$A(x) = Q(x) \cdot (x-k).$$

## Ruffini's theorem

$A(x)$ can be divided by $(x-k)$ with $R(x) = 0$ if and only if $A(k) = 0$, where $k$ is called **root** of $A(x)$.

# Factorization with polynomial division (Ruffini's theorem) II

1. Write the **divisors** of $a_p$ (the integer that dividing $a_p$ gives an integer)
2. Find the divisor $k$ such that $A(k) = 0$, that is called the **root** or **zero** of the polynomial. The Ruffini's theorem assures that $A(x)$ can be divided by $(x - k)$ with $R(x) = 0$.
3. Divide $A(x)$ by $(x - k)$ (ATTENTION: $k$ has changed sign) using the polynomial division by column.
4. Obtain the quotient $Q(x)$ and $R = 0$ (if $R \neq 0$, it is wrong the choice of the divisor $k$)
5. $A(x) = Q(x)(x - k)$.

# Example

**Simplify** $A(x) = 5x^2 - 4x - 1$.

- Write the divisors of $-1$: $\{-1, +1\}$
- Find for which $k \in \{-1, +1\}$ gives $A(k) = 0$:

$$A(-1) = 5 + 4 - 1 = 8, \quad \text{it is not correct}$$

$$A(+1) = 5 - 4 - 1 = 0, \quad \text{it is correct}$$

- Our $k$ is $+1$. We divide $(5x^2 - 4x - 1)$ by $(x - 1)$

$$\begin{array}{ccc|l} 5x^2 & -4x & -1 & x-1 \\ \cline{4-4} -5x^2 & +5x & & = 5x+1 \\ \hline 0 & x & -1 & \\ & -x & +1 & \\ \cline{2-3} & & 0 & \end{array}$$

- $R = 0$, thus it is correct

$$(5x^2 - 4x - 1) = (x - 1)(5x + 1).$$

# Exercises: factorize the following expression with polynomial division

1. $2x^2 + 3x - 2$
2. $2x^3 + 3x^2 - 17x - 30$
3. $x^5 - 4^4 - 10x^3 - 8x^2$

# Algebraic Fractions I

An algebraic fraction is a fraction where the numerator and/or the denominator are polynomials.

**Examples:**

- $\frac{x+2}{x-1}$
- $\frac{3x^2-2x+1}{x^2-1}$

**Operations:**

1. **Simplification:** Factorize the numerator and the denominator and cancel out common factors from the numerator and the denominator.
   **Example:**

$$\frac{2x^2-8}{4x(x-2)} = \frac{2(x^2-4)}{4x(x-2)} = \frac{2(x-2)(x+2)}{4x(x-2)} = \frac{(x-2)(x+2)}{2x}$$

# Algebraic Fractions II

2. **Algebraic sum:** find the LCD and convert each fraction to have the common denominator
**Example:**
$$\frac{1}{x}+\frac{1}{x+1}=\frac{x+1+x}{x(x+1)}=\frac{2x+1}{x(x+1)}$$
3. **Multiplication:** multiply the numerators together and the denominators together.
**Example:**
$$\frac{x}{x+2}\cdot\frac{x+1}{x-1}=\frac{x(x+1)}{(x+2)(x-1)}$$
4. **Division:** multiply by the reciprocal of the divisor.
**Example:**
$$\frac{x}{x+2}\div\frac{x+1}{x-1}=\frac{x}{x+2}\cdot\frac{x-1}{x+1}=\frac{x(x-1)}{(x+2)(x+1)}$$

# Example

$$\frac{\frac{1}{2x}-\frac{1}{x^2}}{\frac{2}{x}-\frac{1}{x-1}}=\frac{\frac{x-2}{2x^2}}{\frac{2(x-1)-x}{x(x-1)}}=\frac{x-2}{2x^2}\div\frac{x-2}{x(x-1)}$$

$$\frac{x-2}{2x^2}\cdot\frac{x(x-1)}{x-2}=\frac{x-1}{2x}$$

# Exercises

1. $\frac{4x^2-9}{2x+3}$
2. $\frac{x^2-1}{x^2+x}$
3. $\frac{x}{x-1} + \frac{x}{x+1}$
4. $\frac{x+2}{x^2-4} \cdot \frac{x-2}{x+1}$
5. $\frac{x^2}{x-1} \div \frac{x^2-1}{x}$

# Common errors

$$
\begin{aligned}
(a+b)^2 &\neq a^2 + b^2 \\
\sqrt{a+b} &\neq \sqrt{a} + \sqrt{b} \\
\frac{1}{a} + \frac{1}{b} &\neq \frac{1}{a+b} \\
\frac{a+b}{a} &\neq b
\end{aligned}
$$

$$
\begin{aligned}
(a \cdot b)^2 &= a^2 \cdot b^2 \\
\sqrt{a \cdot b} &= \sqrt{a} + \sqrt{b} \\
\frac{1}{a} \cdot \frac{1}{b} &= \frac{1}{a \cdot b}
\end{aligned}
$$