Lecture 2
Powers and roots

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Lecture 2

1. Powers

2. n-th root

Powers

Let us consider $a \in \mathbb{R}$ and $n \in \mathbb{N}$, then *"a at the power n"* is

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$$

- a is the base
- n is the exponent

Example:

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 9 \cdot 3 \cdot 3 = 27 \cdot 3 = 81$$

$$\left(\frac{1}{2}\right)^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

Basic Properties of Powers

For every real number $a \in \mathbb{R}$,

1. $a^1 = a$
2. for $a \neq 0$, $a^0 = 1$
3. for $a \neq 0$ and $n \in \mathbb{Z}$,
$$a^{-n} = \frac{1}{a^n}.$$
For example:
$$(-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{9} = \frac{1}{9}$$

Further Properties

1. (same base)

$$a^r \cdot a^s = a^{r+s} \quad \text{and} \quad \frac{a^r}{a^s} = a^r \frac{1}{a^s} = a^r a^{-s} = a^{r-s}$$

2. (same exponent)

$$a^r b^r = (ab)^r \quad \text{and} \quad \left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$$

3. (power of power)

$$(a^r)^s = a^{r \cdot s}$$

Note

Be careful! $(a+b)^r \neq a^r + b^r$. For example:

$$(2+3)^2 = 5^2 = 25 \quad \text{while} \quad 2^2 + 3^2 = 4 + 9 = 13$$

n-th root

The n-th root of a is

$$\sqrt[n]{a}.$$

- n is the index (or degree), with $n \in \mathbb{N}$ and $n \neq 0$
- $a \in \mathbb{R}$ is the radicand

1. For n even, $\sqrt[n]{a}$ is well-defined for $a \geq 0$ and $\sqrt[n]{a} = b \geq 0$ such that $b^n = a$;
2. for n odd, $\sqrt[n]{a}$ is well-defined for every $a \in \mathbb{R}$ and $\sqrt[n]{a} = b \in \mathbb{R}$ such that $b^n = a$.

n-th root: properties

Properties for $a > 0$:

1. $\sqrt[n]{a^n} = a$
2. $\sqrt[n]{a^p} \cdot \sqrt[n]{a^q} = \sqrt[n]{a^p \cdot a^q} = \sqrt[n]{a^{p+q}}$
   In general, if they don't have the same index, the multiplication cannot be done. We'll see it later...
3. $(\sqrt[n]{a})^p = \sqrt[n]{a^p}$
4. $\sqrt[n]{\sqrt[m]{a}} = \sqrt[n \cdot m]{a}$

Note that if $a \leq 0$ the previous properties don't hold, e.g.,

$$\sqrt[4]{(-4)(-3)} = \sqrt[4]{12} \neq \sqrt[4]{-4}\sqrt[4]{-3}$$

Simplification I

Simplification techniques:

- Transport **inside** the root: raise that quantity to the index of the root

$$3\sqrt{2} = \sqrt{3^2} \cdot \sqrt{2} = \sqrt{3^2 \cdot 2}$$

- Transport **outside** the root: decompose that quantity in order to have (at least one of) its factors whit an exponent that is a multiple of the index of the root. This factor will be carried out by dividing its exponent by the index of the root.

Note

Addition (and subtraction) of roots cannot be simplified, i.e.,

$$\sqrt{3} + \sqrt{5} \neq \sqrt{3+5}$$

Simplification II

$$\sqrt[3]{8} = \sqrt[3]{2^3} = 2$$

$$\sqrt[3]{\frac{1}{1000}} = \sqrt[3]{\frac{1}{10^3}} = \frac{\sqrt[3]{1}}{\sqrt[3]{10^3}} = \frac{1}{10}$$

$$\sqrt[3]{-125} = \sqrt[3]{-5^3} = -5$$

$$\sqrt[3]{16} = \sqrt[3]{2^4} = \sqrt[3]{2^3 \cdot 2} = \sqrt[3]{2^3} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}$$

Rationalization I

The root rationalization is the process by which the radical in the denominator of a fraction is eliminated.
Procedure: multiply and divide for the root at the denominator and solve

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \cdot \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{|b|}.$$

Example:

$$\frac{2}{\sqrt{7}} = \frac{2}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{2\sqrt{7}}{7}.$$

Rationalization II

If the denominator is composed of a sum/difference, it is necessary to use the rules of **special products** (we'll see them later).
For example,

$$\frac{1}{\sqrt{2}-\sqrt{5}} = \frac{1}{\sqrt{2}-\sqrt{5}} \cdot \frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}+\sqrt{5}} = \frac{\sqrt{2}+\sqrt{5}}{(\sqrt{2})^2-(\sqrt{5})^2}$$

$$= \frac{\sqrt{2}+\sqrt{5}}{2-5} = \frac{\sqrt{2}+\sqrt{5}}{-3}.$$

Powers with rational exponent

The power of $a \in \mathbb{R}$, with $a \geq 0$, with exponent that is given by a **rational number** $\frac{n}{d} \in \mathbb{Q}$ is the d-th root of a^n, that is

$$a^{\frac{n}{d}} = \sqrt[d]{a^n}, \quad a \geq 0$$

Examples:

- $5^{\frac{2}{3}} = \sqrt[3]{5^2}$
- $(-1)^{\frac{1}{2}}$ does not exists
- $3^{-\frac{1}{2}} = \frac{1}{3^{\frac{1}{2}}} = \frac{1}{\sqrt[2]{3}}$

Operations with roots

Sum and difference

- We can add or subtract only **like radicals**, i.e., radicals with the same index and the same radicand:

$$a\sqrt[n]{b} + c\sqrt[n]{b} = (a+c)\sqrt[n]{b}.$$

Example:

$$2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3}.$$

- In general:

$$\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}.$$

Product

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$

Example:

$$\sqrt{6} \cdot \sqrt{24} = \sqrt{144} = 12.$$

Attention!

Sum and difference: simplify first!

$$\sqrt{12}+\sqrt{27}=\sqrt{4\cdot 3}+\sqrt{9\cdot 3}=2\sqrt{3}+3\sqrt{3}=5\sqrt{3}.$$

Key idea

Two radicals can be added or subtracted only after they have been reduced to the **same radical**.

Operations with roots

Division

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}, \qquad b \neq 0.$$

Example:

$$\frac{\sqrt{18}}{\sqrt{2}} = \sqrt{\frac{18}{2}} = \sqrt{9} = 3.$$

Important

The rules for multiplication and division apply when the roots have the **same index**.

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}, \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

But:

$$\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}, \qquad \sqrt{a} - \sqrt{b} \neq \sqrt{a-b}.$$

Product of roots with different indices

Remember that we can rewrite each root as a **fractional power**: $\sqrt[n]{a} = a^{\frac{1}{n}}$. Then we use the properties of powers.

Example:

$$\sqrt{2} \cdot \sqrt[3]{4} = 2^{\frac{1}{2}} \cdot 4^{\frac{1}{3}}$$

Since $4 = 2^2$,

$$= 2^{\frac{1}{2}} \cdot (2^2)^{\frac{1}{3}} = 2^{\frac{1}{2}} \cdot 2^{\frac{2}{3}}$$

(power with same base)

$$= 2^{\frac{1}{2}+\frac{2}{3}} = 2^{\frac{7}{6}} = 2\sqrt[6]{2}.$$

Product of roots with different indices

Example:

$$\sqrt[3]{12} \cdot \sqrt[6]{18}$$

Rewrite as fractional powers:

$$12^{\frac{1}{3}} \cdot 18^{\frac{1}{6}}.$$

Factorize the bases:

$$(2^2 \cdot 3)^{\frac{1}{3}} (2 \cdot 3^2)^{\frac{1}{6}}$$

$$= 2^{\frac{2}{3}} 3^{\frac{1}{3}} \cdot 2^{\frac{1}{6}} 3^{\frac{2}{6}}$$

$$= 2^{\frac{5}{6}} 3^{\frac{2}{3}}.$$

It cannot be simplified more!