Lecture 1
Set Theory and Numbers

Preliminary course - Mathematics 2026/2027
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Lecture 1

1. Set and subset
2. Operations on sets
3. Number sets

Set

Definition

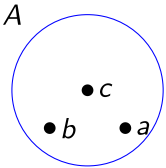
A collection of objects is a set if there is an **objective law/criteria** that allows us to identify the elements of the set unambiguously. Otherwise, a set can be identified by explaining **all** its elements.

For example:

- *"The most beautiful cities in Italy"* is not a set!
- *"The cities with more that 1 million inhabitants"* is a set

Representation of a set

- **Graph representation:** Venn diagram



- **Roster Notation**: by listing all its elements between braces

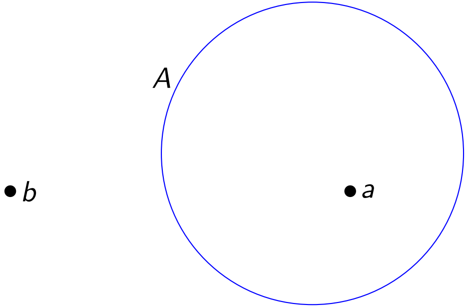
$$A = \{1, 2, 10, 100\}$$

- **Set-Builder Notation**: a property that all members of a set have to satisfy

$$A = \{x \text{ is a number } | x \text{ is odd } \}$$

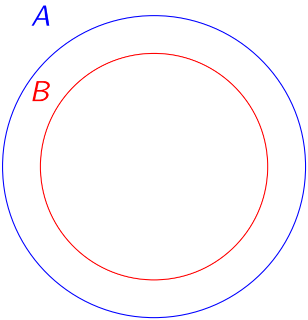
Notation

- $A, B, \ldots$: sets,
- $a, b, c, \ldots$: elements
- $a \in A$: *a is a member of A*
- $b \notin A$: *a is* **not** *a member of A*
- $\emptyset$: emptyset
- $|A|$: cardinality of a set, i.e., the number of elements in A



Subset

Given two sets A and B,



- if every element of B is also an element of A, we say $B \subseteq A$ (i.e., B is a **subset** of A),

$$\forall b \in B \Rightarrow b \in A;$$

- if there exists an element in A that does not belong to B, then we say $B \subset A$ (i.e., B is a **proper subset** of A)

$$\forall b \in B \Rightarrow b \in A,$$

$$\text{and } \exists a \in A \text{ such that } a \notin B.$$

Note

- For every A it holds that $A \subseteq A$
- if $A \subseteq B$ and $B \subseteq C$, then $A = B$
- for every A, it holds that $\emptyset \subseteq A$
- **Member and subset**

$$A = \{Months\ of\ the\ year\}$$

$$\{june\} \subset A, \quad \text{"june" is a set}$$

$$\text{june} \in A, \quad \text{"june" is a member}$$

- **Quantifiers:**
  - $\forall a \in A$: for all a that is a member of the set A, it holds that...
  - $\exists b \in B$: it exists (at least) an element b that is a member of the set B, such that...

Operations on Sets: Intersection I

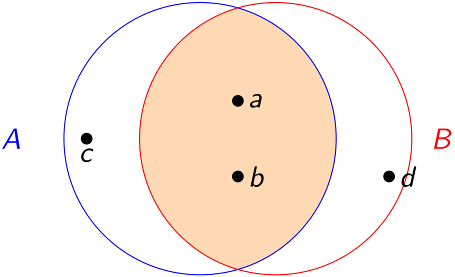
Definition

The intersection of two sets A and B is the set composed by elements that are in both A and B, only:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Example:

$$A = \{a, b, c\}, \quad B = \{a, b, d\}, \quad A \cap B = \{a, b\}$$



Properties

- $A \cap \emptyset = \emptyset$;
- (commutative) $A \cap B = B \cap A$;
- (associative) $A \cap (B \cap C) = (A \cap B) \cap C$.

Operations on Sets: Intersection II

If $A \cap B = \emptyset$, A and B are said to be **disjoint**
If $A \subseteq B$, then $A \cap B = A$

Example:

$$A = \{1, 2, 5, 9\}, \quad B = \{2, 4, 10\}, \quad C = \{3, 4\}$$

It holds that $A \cap B = \{2\}$, $A \cap C = \emptyset$ and $B \cap C = \{4\}$. Thus A and C are disjoint sets.

Operations on Sets: Union

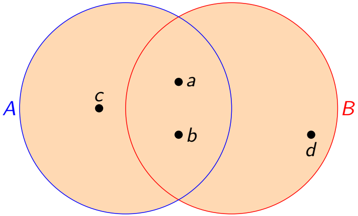
Definition

The union of two sets A and B is the set of all the elements in A and in B without repeating the same elements:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Example:

$$A = \{a, b, c\}, \quad B = \{a, b, d\}, \quad A \cup B = \{a, b, c, d\}$$



Properties

- $A \cup \emptyset = A$;
- (commutative) $A \cup B = B \cup A$;
- (associative) $A \cup (B \cup C) = (A \cup B) \cup C$;
- (distributive) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Operations on Sets: Difference

Definition

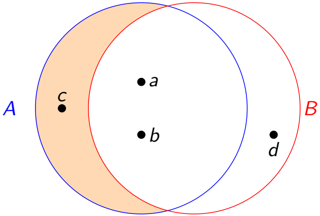
The difference of two sets A and B is the set of elements in A that are not in B:

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Defining the **complement set** of B as $B^c = \{x | x \notin B\}$, it holds that $A \setminus B = A \cap B^c$.

Example:

$$A = \{a, b, c\}, \quad B = \{a, b, d\}, \quad A \setminus B = \{c\}$$



Note:

$A \backslash B \neq B \backslash A.$

Complement

Definition

The complement of a set A (with respect to the **universe set** U) is the set of all the elements not in A:

$$A^c = \{x \in U \mid x \notin A\} = U \setminus A$$

If we have A and B such that $B \subseteq A$, the complement of B is the set of elements that are not in B and it is

$$B_A^c = \{x \in A \mid x \notin B\} = A \setminus B.$$

Example:

$$A = \{a, b, c, d\}, \quad B = \{a, b\},$$

It holds that $B_A^c = \{c, d\}$.

Cartesian product

Given two sets A and B, the **cartesian product** $A \times B$ is the set of ordered couples where the first element of the couple is a member of A and the second element of the couple is an element of B.

$$A \times B = \{(a, b),\ a \in A,\ b \in B\}.$$

Example: $A = \{1, 2\}$, $B = \{x, y\}$.

$$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}.$$

Note that:

- $B \times A = \{(x, 1), (x, 2), (y, 1), (y, 2)\} \neq A \times B$;
- $A \times A = A^2 = \{(a, a), (a, b), (b, a), (b, b)\}$.

Power set

Definition

The power set of a set A is the set of all subsets of A, including the emptyset and A itself. It is denoted with

$$\mathcal{P}(A) = 2^A.$$

Example:

$$A = \{a, b, c\}$$

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}.$$

It holds that $|\mathcal{P}(A)| = 2^n$ where $n = |A|$.

Exercise 1

1. Define set A as the set of even numbers between 1 and 10.
2. Define set B as the set of odd numbers between 1 and 10.
3. Find the cardinality of set A.
4. Find the cardinality of set B.
5. Find the intersection of A and B: $A \cap B$.
6. Consider the set $C = \{1, 5, 9\}$
7. Find the intersection of B and C: $B \cap C$.
8. Find the union of A and C: $A \cup C$.
9. Find the difference between A and B: $A \setminus B$.
10. Find the difference between C and B: $C \setminus B$.
11. Verify if set C is a subset of A.
12. Verify if set $D = \{1, 3, 5\}$ is a subset of B.

Exercise 1 – Solutions

1. $A = \{2, 4, 6, 8, 10\}$
2. $B = \{1, 3, 5, 7, 9\}$
3. $|A| = 5$
4. $|B| = 5$
5. $A \cap B = \varnothing$
6. $C = \{1, 5, 9\}$
7. $B \cap C = \{1, 5, 9\}$
8. $A \cup C = \{1, 2, 4, 5, 6, 8, 9, 10\}$
9. $A \setminus B = \{2, 4, 6, 8, 10\}$
10. $C \setminus B = \varnothing$
11. $C \subseteq B \quad \checkmark$
12. $D = \{1, 3, 5\} \subseteq B \quad \checkmark$

Exercise 2

Consider the sets

$$A = \{1, 9, 22, -5, 3, -11\}$$

$$B = \{-3, 2, 1, 3, 20\}$$

Check is the following sentences are true or false

1. $-5 \in A$
2. $3 \subset A$
3. $\{3, 1\} \subseteq A$
4. $-5 \in A \cap B$
5. $-11 \in A \cup B$
6. $(-3, 3) \in A \times B$
7. $(-3, 3) \in B \times A$
8. $(1, 2) \in A \times B$
9. $(1, 2) \in B \times A$

Exercise 2 – Solutions

1. $-5 \in A$ True
2. $3 \subset A$ False (3 is an element, not a set)
3. $\{3, 1\} \subseteq A$ True
4. $-5 \in A \cap B$ False $(-5 \notin B)$
5. $-11 \in A \cup B$ True
6. $(-3, 3) \in A \times B$ False $(-3 \notin A)$
7. $(-3, 3) \in B \times A$ True
8. $(1, 2) \in A \times B$ True
9. $(1, 2) \in B \times A$ False $(2 \notin A)$

Number Sets: Natural Numbers I

Definition

The set of natural numbers is the set of all positive whole numbers including the zero:

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

Properties of **addition**:

- Commutative: $\forall a, b \in \mathbb{N}, a + b = b + a$
- Associative: $\forall a, b, c \in \mathbb{N}, a + (b + c) = (a + b) + c$
- Neutral element: $\exists 0 \in \mathbb{N} \mid \forall a \in \mathbb{N}, a + 0 = a$.

Properties of **multiplication**:

- Commutative: $\forall a, b \in \mathbb{N}, a \cdot b = b \cdot a$
- Associative: $\forall a, b, c \in \mathbb{N}, a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- Distributive: $\forall a, b, c \in \mathbb{N},\ a \cdot (b + c) = a \cdot b + a \cdot c$
- Neutral element: $\exists 1 \in \mathbb{N} \mid \forall a \in \mathbb{N}, a \cdot 1 = a$.

Number Sets: Natural Numbers II

Note

$\mathbb{N}$ is closed under addition and multiplication: for any $n, m \in \mathbb{N}$:

- $n + m \in \mathbb{N}$
- $n \cdot m \in \mathbb{N}$

On the contrary, it is **not closed** with respect to the subtraction and division:

- $2 - 3 = -1 \notin \mathbb{N}$.
- $5 : 6 = \frac{5}{6} \notin \mathbb{N}$.

It does not exist the **opposite**, that is an element $b \in \mathbb{N}$, such that $a + b = 0 \implies$ Integer numbers

Number Sets: Integers I

Definition

The set of integer numbers is the set of all natural numbers together with their opposites:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}, \qquad \mathbb{N} \subset \mathbb{Z}$$

Properties of **addition** and **multiplication** as $\mathbb{N}$, and

- Existence of the opposite: $\forall a \in \mathbb{Z}, \exists -a \in \mathbb{Z} \mid a + (-a) = 0$

Recall that

$$(+)\cdot(+) = (-)\cdot(-) = +,$$
$$(+)\cdot(-) = (-)\cdot(+) = -.$$

Number Sets: Integers II

Note

$\mathbb{Z}$ is closed under addition, multiplication and difference: for any $n, m \in \mathbb{Z}$:

- $n \pm m \in \mathbb{Z}$
- $n \cdot m \in \mathbb{Z}$

However, the set $\mathbb{Z}$ is **not closed** under the division: $2 \div 3 = \frac{2}{3} \notin \mathbb{Z}$
$\implies$ Rational numbers

Number Sets: Rational Numbers I

All fractions of integer numbers, with denominator not zero

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, \text{ with } q \neq 0 \right\}, \qquad \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$$

Properties of addition and multiplication as $\mathbb{N}$, and

- Existence of the opposite: $\forall a \in \mathbb{Q}, \exists -a \in \mathbb{Q} \mid a + (-a) = 0$
- Existence of the reciprocal: $\forall a \in \mathbb{Q}, a \neq 0, \exists \frac{1}{a} \in \mathbb{Q} \mid a \cdot \frac{1}{a} = 1$

Note

Rational numbers can be:

- Integers: $\frac{0}{100} = 0, \frac{8}{4} = 2, \frac{-10}{2} = -5, \ldots$
- Decimals with **finite** representation: $\frac{1}{8} = 0.125, \ldots$
- Decimals with **infinite** periodic representation: $\frac{1}{3} = 0.3333\ldots = 0.\overline{3}$.

Number Sets: Rational Numbers II

$\frac{p}{q} \in \mathbb{Q}$ is the result of the division between p and q:

- $\frac{0}{q} = 0$;
- $\frac{p}{0}$ is impossible: which is the number that multiplied by 0 gives p?
- $\frac{0}{0}$ is indeterminate

Note

Decimals with infinite non-periodic representation are not rational numbers, for instance,

$1.414213562373095\ldots = \sqrt{2}$ (length of hypotenuse of triangle with legs of 1)

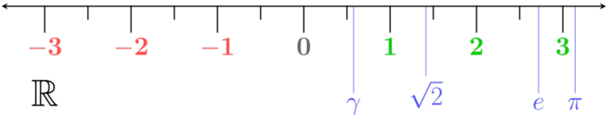
$3.141592653589793\ldots = \pi$ (ratio of a circle's circumference to its diameter)

These numbers are called **irrational numbers** and they are not in $\mathbb{Z}$.

Number Sets: Real numbers I

Set of all rational and irrational numbers

$$\mathbb{R} = \mathbb{Q} \cup \{\text{Irrational}\}, \qquad \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Properties of addition and multiplication as $\mathbb{Q}$.

Subsets of $\mathbb{R}$

Intervals

- Closed: $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$
- Open: $(a, b) =]a, b[= \{x \in \mathbb{R} | a < x < b\}$
- Right open: $[a, b) = [a, b[= \{x \in \mathbb{R} | a \leq x < b\}$
- Left open: $(a, b] =]a, b] = \{x \in \mathbb{R} | a < x \leq b\}$

Unbounded intervals

- $[a, +\infty) = [a, +\infty[= \{x \in \mathbb{R} | x \geq a\}$,
- $(a, +\infty) =]a, +\infty[= \{x \in \mathbb{R} | x > a\}$,
- $(-\infty, a] =] + \infty, a] = \{x \in \mathbb{R} | x \leq a\}$,
- $(-\infty, a) =] + \infty, a[= \{x \in \mathbb{R} | x < a\}$.