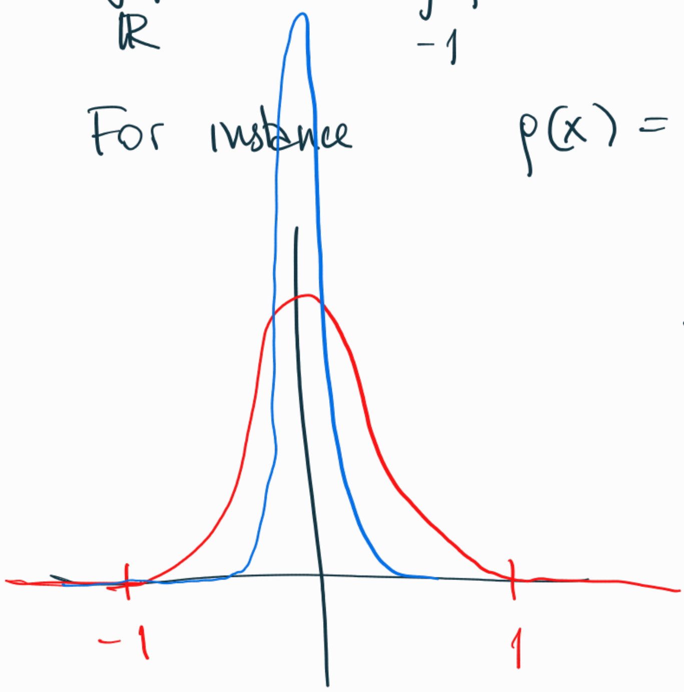
Mollifiers

Let $\rho_n(x)$ be a sequence of mollifiers, i.e.
we take a function $\rho(x) \in C_c^\infty(\mathbb{R})$, $\operatorname{supp} \rho \subset [-1,1]$, $\rho \geq 0$.

$$\int_{\mathbb{R}} \rho(x)\,dx = \int_{-1}^{1} \rho(x)\,dx = 1$$

For instance
$$\rho(x) = \begin{cases} 0 & |x| \geq 1 \\ c\, e^{-\frac{1}{1-x^2}} & |x| < 1 \end{cases}$$



Then we take
$$\rho_n(x) = n\, \rho(nx)$$

THEOREM If $f \in L^p(\mathbb{R})$ $\quad 1 \leq p < \infty$.
then
$$\underbrace{\rho_n * f}_{\in\, C^\infty(\mathbb{R})} \longrightarrow f \quad \text{in } L^p(\mathbb{R}).$$

PROP If $f \in C_c(\mathbb{R})$ (continuous with compact support)
then $\rho_n * f \to f$ in $L^p(\mathbb{R})$.

Proof by Heine–Cantor's theorem, f is uniformly continuous, i.e.
$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x, y \in \mathbb{R}$ verifying $|x-y| < \delta$
$$\Rightarrow |f(x) - f(y)| < \varepsilon$$

We prove that $(\rho_n * f)(x) \to f(x) \quad \forall x \in \mathbb{R}$.

$$|(\rho_n * f)(x) - f(x)| = \left| \int_{\mathbb{R}} f(x-y)\rho_n(y)\,dy - \underbrace{\int_{\mathbb{R}} \rho_n(y)\,dy}_{=1} f(x) \right| =$$

$$= \left| \int_{\mathbb{R}} [f(x-y) - f(x)]\, \rho_n(y)\,dy \right| \leq$$

$$\leq \int_{\mathbb{R}} \underbrace{|f(x-y) - f(x)|}_{\wedge\ \varepsilon} \underbrace{\rho_n(y)}_{\neq 0 \text{ only on } [-\frac{1}{n}, \frac{1}{n}]} dy <$$

For fixed ε, I choose n such that $\frac{1}{n} < \delta$ ← the one from uniform continuity

$$< \varepsilon \underbrace{\int_{\mathbb{R}} \rho_n(y)\,dy}_{= 1} = \varepsilon$$

These computations are independent on $x \Rightarrow$ we have proved that
$\forall \varepsilon \quad \exists \bar{n} \quad$ s.t. $\forall n > \bar{n}$

$$|(\rho_n * f)(x) - f(x)| < \varepsilon \qquad \forall x \in \mathbb{R}$$

$$\Rightarrow \sup_{x \in \mathbb{R}} |(\rho_n * f)(x) - f(x)| \leq \varepsilon \qquad \forall n > \bar{n}$$

That is, we have uniform convergence, which is the same as L^∞-convergence. Moreover $\rho_n * f - f$ is zero outside of $\operatorname{supp} f + [-\frac{1}{n}, \frac{1}{n}]$ (bounded set)
$\Rightarrow$ convergence in L^p $\forall p$. $\square$

PROP 2 $\forall f \in L^p(\mathbb{R})$ $\quad 1 \le p < \infty$

$\exists\ f_1 \in C_c(\mathbb{R})$ s.t. $\quad \|f - f_1\|_p < \varepsilon$

(proved last time in the case $p = 1$).

Proof of Theorem

If $f \in L^p(\mathbb{R})$, and $\varepsilon > 0 \overset{\text{Prop 2}}{\Longrightarrow} \exists\ f_1 \in C_c(\mathbb{R})$ s.t.

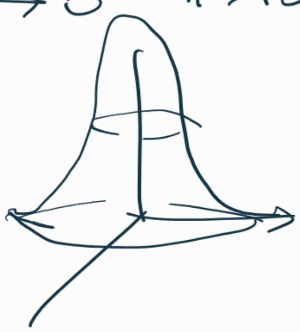
$\|f - f_1\|_p < \varepsilon$.

$$\|(\rho_n * f) - f\|_p \le \underbrace{\|(\rho_n * f) - (\rho_n * f_1)\|_p}_{\substack{= \\ \|\rho_n * (f - f_1)\|_p \\ \wedge\!| \\ \underbrace{\|\rho_n\|_1}_{=\,1}\ \underbrace{\|f_1 - f\|_p}_{<\,\varepsilon}}} + \|(\rho_n * f_1) - f_1\|_p + \underbrace{\|f_1 - f\|_p}_{<\,\varepsilon}$$

$$< 2\varepsilon + \underbrace{\|(\rho_n * f_1) - f_1\|_p}_{\to\, 0 \text{ by Prop 1.}}$$

Since ε is arbitrary, we have proved that

$$\|(\rho_n * f) - f\|_p \to 0 \quad n \to +\infty \quad \square$$



All this works in $\mathbb{R}^2$, $\mathbb{R}^3$, etc....

except that

$$\rho_n(x) = n^2 \rho(nx) \quad \text{in } \mathbb{R}^2$$

$$n^3 \rho(nx) \quad \text{in } \mathbb{R}^3$$

Eigenvalues and eigenfunctions of the Laplacian.

$\Omega \subset \mathbb{R}^n$ open bounded set.

Consider this problem

$$(P) \quad \begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

We will say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (P) (more precisely, an eigenvalue of the laplacian on Ω with homogeneous Dirichlet boundary conditions) if there exists a function $u \not\equiv 0$ solution of problem (P)

u will be called an eigenvector (or an eigenfunction) relative to λ. Of course any multiple of u is also an eigenfunction.

This is a more complicated and differential version of the classical eigenvalue problem for matrices, i.e.

$$\underset{\uparrow \text{matrix}}{A} v = \lambda v, \quad \text{where } A \text{ is a symmetric matrix}$$

Why is this problem useful?

Imagine that we want to study the heat equation in a domain. $u(x,t) \quad x \in \Omega \subset \mathbb{R}^N$

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0,+\infty) \\ u(x,0) = u_0(x) \ \text{given} & \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty) \end{cases}$$

We seek a solution by separation of variables

$$u(x,t) = X(x)\,T(t)$$

The equation becomes

$$u_t = \Delta_x u \iff X(x)\,T'(t) = T(t)\,\Delta X(x)$$

$$\iff \frac{T'(t)}{T(t)} = \frac{\Delta X(x)}{X(x)} = \text{constant} = -\lambda$$

$$\Rightarrow \begin{cases} -\Delta X(x) = \lambda X(x) & \text{on } \Omega. \\ X = 0 & \text{on } \partial\Omega \end{cases}$$

We consider the eigenvalue problem

$$(P) \begin{cases} -\Delta u = \lambda u & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense of weak solutions.

Def λ, u are eigen$\begin{cases}\text{value} \\ \text{function}\end{cases}$ of problem (P).

if $\lambda \in \mathbb{R}$ and $u \in H_0^1(\Omega)$ s.t.

$$(*) \quad \int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega u \cdot v \, dx \quad \forall v \in H_0^1(\Omega)$$

Actually, if Ω has smooth boundary, one can prove that all eigenfunctions are also classical solutions

If we take $v = u$ in $(*)$, we obtain

$$\int_\Omega |\nabla u|^2 dx = \lambda \int_\Omega u^2 dx$$

Remember the Poincaré inequality:

$$\int_\Omega u^2 dx \leq c(\Omega) \int_\Omega |\nabla u|^2 dx \quad \forall u \in H^1_0(\Omega)$$

so we obtain

$$\int_\Omega u^2 dx \leq c(\Omega) \int_\Omega |\nabla u|^2 = c(\Omega)\lambda \int_\Omega u^2 dx$$

$$\Rightarrow \quad c(\Omega)\lambda \geq 1 \quad \Rightarrow \quad \lambda \geq \frac{1}{c(\Omega)} > 0$$

we will show later that here is =

$\Rightarrow$ All eigenvalues are strictly positive

<u>THEOREM</u> (Spectral theorem for the Laplacian).

$\exists$ a sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_k \leq \dots$$

s.t. $\lambda_k \longrightarrow +\infty$ as $k \to +\infty$

$\exists$ $\{e_k(x)\}$ of eigenfunctions relative to λ_k s.t.

$\{e_k\}$ constitute an orthonormal basis of L^2

(i.e. a maximal orthonormal system in $L^2(\Omega)$)

$\Rightarrow$ It means that we can do Fourier series

$$u = \sum_{k=1}^{\infty} \underbrace{(u, e_k)_{L^2}}_{\hat{u}_k} e_k$$

$$\Longrightarrow \quad \|u\|^2_{L^2} = \sum_{k=1}^{\infty} \hat{u}_k^2 \quad \text{Parseval's inequality.}$$

This implies that the functions $\left\{\frac{e_k}{\sqrt{\lambda_k}}\right\}$ make an orthonormal base in $H_0^1(\Omega)$, with the scalar product

$$(u,v)_{H_0^1} = \int_\Omega \nabla u \cdot \nabla v \, dx$$

Let us check that $\left\{\frac{e_k}{\sqrt{\lambda_k}}\right\}$ are orthonormal

the e_k satisfy $\quad -\Delta e_k = \lambda_k e_k \quad \times e_j$

$$\int_\Omega \nabla e_k \cdot \nabla e_j = \lambda_k \int_\Omega e_k \cdot e_j = \begin{cases} 0 & \text{if } j \neq k \\ \lambda_k & \text{if } j = k \end{cases}$$

$$\Rightarrow \left(\frac{e_k}{\sqrt{\lambda_k}}, \frac{e_j}{\sqrt{\lambda_j}}\right)_{H_0^1(\Omega)} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

It is easy to check that this system is maximal.

A trivial case: dimension 1.

$$\Omega = (0, \pi)$$

$$(P) \begin{cases} -u'' = \lambda u \\ u(0) = u(\pi) = 0 \end{cases}$$

$\lambda < 0 \Rightarrow u \equiv 0$

$\lambda = 0 \Rightarrow u \equiv 0$

$\lambda > 0 \Rightarrow u(x) = A \sin(\sqrt{\lambda}\, x) + \cancel{B \cos(\sqrt{\lambda}\, x)}$

$u(0) = 0 \Rightarrow 0 = B$

$u(\pi) = 0 \Rightarrow 0 = A \sin(\sqrt{\lambda}\, \pi)$ (but $A \neq 0$)

$\Rightarrow \quad \sin(\sqrt{\lambda}\pi) = 0 \quad \Longrightarrow \quad \sqrt{\lambda} = k$

$\Rightarrow \quad \lambda = k^2$

so, the eigenvalues are $\lambda_k = k^2$ $\quad k \in \mathbb{N}_+$

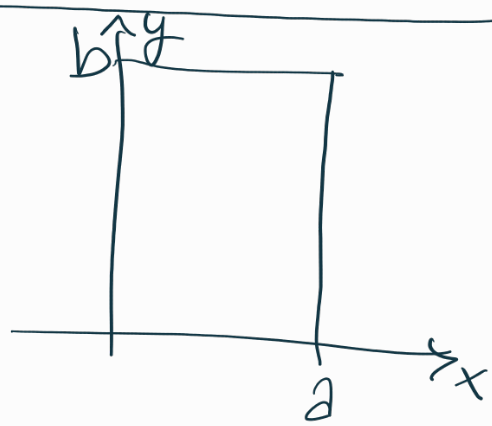
$$e_k = \frac{2}{\pi} \sin(kx)$$

If the interval is $(0, L)$ $\quad L > 0.$

$$\Rightarrow \quad \lambda_k = \left(\frac{k\pi}{L}\right)^2 \qquad k = 1, 2, 3 \ldots$$

$$e_k(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi x}{L}\right)$$

$N = 2, \quad \Omega = (0, a) \times (0, b)$



$u(x) = X(x)\, Y(y)$

$-\Delta u = \lambda u$

$\Rightarrow -\Delta X \cdot Y - \Delta Y \cdot X = \lambda X Y$

$\Rightarrow -\dfrac{\Delta X(x)}{X(x)} - \dfrac{\Delta Y(y)}{Y(y)} = \lambda \quad \Rightarrow$ both ratios are constant

$$\Rightarrow \lambda_{m,n} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

$$e_{m,n}(x, y) = C_{m,n} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \qquad m, n \geq 1$$

Moreover, it can be proved that the first eigenvalue λ_1 is isolated, and has a unique eigenfunction u_1, which is strictly positive.

Heat equation

$$\begin{cases} u_t = \Delta u & (x,t) \in \Omega \times (0,+\infty) \\ u(x,0) = u_0(x) & x \in \Omega \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,+\infty). \end{cases}$$

We saw that

$u(x,t) = T(t)\,X(x)$ is a sol^n if

$$\frac{T'(t)}{T(t)} = \frac{\Delta X(x)}{X(x)} = -\lambda$$

possible only if $\lambda = \lambda_k > 0$

$$X_k(x) = c_k\, e_k(x) \qquad T_k'(t) = -\lambda_k T_k(t)$$

$$\Rightarrow T_k(t) = e^{-\lambda_k t}$$

I can sum over k.

$$u(x,t) = \sum_{k=1}^{\infty} c_k\, e_k(x)\, e^{-\lambda_k t}$$

this satisfies $u(x,t) = 0$ if $x \in \partial\Omega$

We have to impose that $u(x,0) = u_0(x)$

$$u(x,0) = \sum_{k=1}^{\infty} c_k\, e_k(x) = u_0(x)$$

If $u_0 \in L^2(\Omega) \Rightarrow c_k = \hat{u}_{0k} = (u_0, e_k)_{L^2}$

Note: All "modes" decay in time for $t \to +\infty$

The one which decays slower is the term with λ_1.

One can prove that $u(x,t) \sim c_1 e_1(x) e^{-\lambda_1 t}$ for $t \to +\infty$

Wave equation

$$\begin{cases} u_{tt} = \Delta u & \text{in } \Omega \times (0,+\infty) \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x) & x \in \Omega \\ u_t(x,0) = v_0(x) & x \in \Omega \end{cases}$$

$$u(x,t) = T(t)\, X(x)$$

$$u_{tt} = \Delta u \iff T''(t)\, X(x) = T(t)\, \Delta X(x)$$

$$\iff \frac{T''(t)}{T(t)} = \frac{\Delta X(x)}{X(x)} = -\lambda_k$$

$\lambda_k > 0$

$$X_k(x) = c_k e_k(x)$$

$$T_k''(t) = -\lambda_k T_k(t)$$

$$T_k(t) = A_k \cos(\sqrt{\lambda_k}\, t) + B_k \sin(\sqrt{\lambda_k}\, t)$$

$$T_k(0) = 1 \implies \boxed{A_k = 1}$$

$$T_k'(t) = -\sqrt{\lambda_k} \sin(\sqrt{\lambda_k}\, t) + B_k \sqrt{\lambda_k} \cos(\sqrt{\lambda_k}\, t)$$

$$\boxed{T_k'(0) = B_k \sqrt{\lambda_k}}$$

$$u(x,t) = \sum_{k=1}^{\infty} c_k T_k(t) e_k(x) \qquad = 0 \text{ on } \partial\Omega$$

Now we impose the initial conditions

$$u(x,0) = \sum_{k=1}^{\infty} c_k e_k(x) = u_0(x)$$

$$\Rightarrow c_k = \hat{u}_{0k} = (u_0, e_k)_{L^2}.$$

$$u_t(x,0) = \sum_{k=1}^{\infty} c_k T_k'(0) e_k(x) =$$

$$= \sum_{k=1}^{\infty} c_k B_k \sqrt{\lambda_k}\, e_k(x) = v_0(x)$$

$\Rightarrow$ I choose B_k such that $c_k B_k \sqrt{\lambda_k} = \hat{v}_{0,k} = (v_0, e_k)$

$$\Rightarrow B_k = \frac{\hat{v}_{0k}}{c_k \sqrt{\lambda_k}}$$

One last result: eigenvalues as minima of functionals

Let $u \in H_0^1(\Omega) \setminus \{0\} \Rightarrow R(u) = \dfrac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}$

Raileigh quotient.

<u>THEOREM</u> $\lambda_1, \lambda_2 \dots \lambda_k \dots$ eigenvalues as before

$$\lambda_1 = \min_{u \in H_0^1(\Omega)\setminus\{0\}} R(u) \quad \text{assumed when } u = e_1$$

$\Rightarrow \displaystyle\int_\Omega u^2 dx \leq \frac{1}{\lambda_1} \int |\nabla u|^2 dx \Rightarrow \frac{1}{\lambda_1}$ is the Poincaré constant

$$\lambda_2 = \min_{\substack{u \in H_0^1(\Omega)\setminus\{0\} \\ u \perp e_1}} R(u) \quad \text{assumed when } u = e_2$$

$$\lambda_k = \min_{\substack{u \in H_0^1(\Omega)\setminus\{0\} \\ u \perp e_1, e_2 \dots e_{k-1}}} R(u) \quad \text{assumed when } u = e_k$$

<u>Proof</u>. $u \in H_0^1(\Omega) \Rightarrow u = \sum_{j=1}^{\infty} \hat{u}_j e_j = \sum_{j=1}^{\infty} (u, e_j) e_j$

$$\Rightarrow \int_\Omega u^2 dx = \|u\|_{L^2}^2 = \sum_{j=1}^{\infty} \hat{u}_j^2$$

$$\int_\Omega |\nabla u|^2 dx = \sum_{j=1}^{\infty} \left[\int_\Omega \nabla u \cdot \frac{\nabla e_j}{\sqrt{\lambda_j}} dx \right]^2 =$$

$$= \sum_j \frac{\left[\int_\Omega \nabla u \cdot \nabla e_j \, dx\right]^2}{\lambda_j} =$$

Remember $\int_\Omega \nabla e_j \cdot \nabla u \, dx = \lambda_j \int_\Omega e_j u \, dx = \lambda_j (e_j, u)_{L^2}$

$$= \lambda_j \hat{u}_j$$

$$= \sum_j \lambda_j \hat{u}_j^2$$

$$\Rightarrow R(u) = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} = \frac{\sum_j \lambda_j \hat{u}_j^2}{\sum_j \hat{u}_j^2}$$

Now, all $\lambda_j \geq \lambda_1$

$$\Rightarrow \quad R(u) \geq \lambda_1 \cancel{\frac{\sum \hat{u}_j^2}{\sum \hat{u}_j^2}} \qquad \forall u \in H_0^1(\Omega)$$

If $u = e_1 \quad \Rightarrow \quad \hat{u}_j = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{if } j > 1 \end{cases}$

$$R(e_1) = \frac{\lambda_1}{1} = \lambda_1$$

If $u \perp e_1 \Rightarrow \hat{u}_1 = 0$

$$\Rightarrow R(u) = \frac{\sum_{j=2}^{\infty} \lambda_j \hat{u}_j^2}{\sum_{j=2}^{\infty} \hat{u}_j^2} \geq \lambda_2$$

$$u = e_2 (\perp e_1) \Rightarrow R(e_2) = \frac{\lambda_2}{1} = \lambda_2$$

... and so on...