Let us consider the following differential problem

$$(P)\begin{cases} -\Delta u = g(u) & \text{in } \Omega \subset \mathbb{R}^N \text{ bounded open set.} \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ Lipschitz - continuous, i.e.

$$|g(s) - g(t)| \leq L\,|s-t| \qquad \forall s,t \in \mathbb{R}.$$

Idea: We fix $v \in L^2(\Omega)$, and we try to solve. $\begin{cases} -\Delta w = g(v) & \\ w = 0 & \text{on } \partial\Omega \end{cases}$

Remember that, if $f \in L^2(\Omega)$, $\exists!$ u weak sol^n of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

i.e. $u \in H^1_0(\Omega) = W^{1,2}_0(\Omega)$ s.t.

$$(*) \quad \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in H^1_0(\Omega)$$

If we take $v = u$ in $(*)$, we obtain.

$$\|\nabla u\|^{\cancel{2}}_{L^2} = \int_\Omega |\nabla u|^2 dx = \int_\Omega f u \, dx \leq \|f\|_{L^2} \|u\|_{L^2} \leq$$

$$\left[\text{Poincaré's inequality: } \|u\|_{L^2(\Omega)} \leq C_p(\Omega, N)\, \|\nabla u\|_{L^2(\Omega)}\right]$$

$$\forall u \in H^1_0(\Omega)$$

$$\leq \|f\|_{L^2} \, c_p \, \cancel{\|\nabla u\|}_{L^2(\Omega)}$$

$$\Rightarrow \boxed{\|\nabla u\|_{L^2} \leq c_p \|f\|_{L^2}}$$

If we take $v \in L^2(\Omega)$, $\Rightarrow g(v) \in L^2(\Omega)$, indeed

$$|g(v)| = |g(0) + g(v) - g(0)| \leq |g(0)| + \underbrace{|g(v) - g(0)|}_{\substack{\wedge\!| \\ L|v|}}$$

$$\leq |g(0)| + L|v| \in L^2(\Omega)$$

$$\Rightarrow |g(v)| \in L^2(\Omega).$$

Therefore I can solve

$$\begin{cases} -\Delta w = g(v) & \text{on } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \qquad (**)$$

$\forall v \in L^2 \quad \exists !\, w \in H^1_0(\Omega) \subseteq L^2(\Omega)$

So I have defined an operator

$$T : L^2(\Omega) \to L^2(\Omega)$$

$$v \longmapsto T(v) = w, \text{ sol}^n \text{ of } (**)$$

I need to find a fixed point for T, i.e.

a function $u \in L^2(\Omega)$ s.t. $T(u) = u$

I will show that under some assumptions, T is a contraction.

i.e.

$$\underbrace{\|T(w_1)}_{v_1} - \underbrace{T(w_2)}_{v_2}\|_{L^2} \leq \alpha \|w_1 - w_2\|_{L^2}, \forall\, w_1, w_2 \in L^2$$

with $\alpha < 1$

$$\left.\begin{array}{l} -\Delta v_1 = g(w_1) \\ -\Delta v_2 = g(w_2) \end{array}\right| \Rightarrow \quad v_1 - v_2 \text{ satisfies}$$

$$-\Delta(v_1 - v_2) = g(w_1) - g(w_2)$$

$$\|T(w_1) - T(w_2)\|_{L^2} = \|v_1 - v_2\|_{L^2} \leq C_p \|\nabla(v_1 - v_2)\|_{L^2}$$

$$\leq C_p^2 \|g(w_1) - g(w_2)\|_{L^2} \leq C_p^2 L \|w_1 - w_2\|_{L^2}$$

$$|g(w_1) - g(w_2)| \leq L|w_1 - w_2|$$

So I have the required estimate with $\alpha = C_p^2 L$

I need $C_p^2 L < 1$.

It can be shown that the Poincare constant satisfies

$$C_p \leq c(N)\, m(\Omega)^{1/N}$$

$$C_p^2 L \leq c(N)^2\, m(\Omega)^{2/N} L < 1$$

if $m(\Omega) < \left(\dfrac{1}{L\, c(N)^2}\right)^{\frac{N}{2}}$

This approach can be used with other fixed point theorems, for instance Schauder's theorem or Schaeffer's theorem.
These results allow to study equations of the form

$$-\Delta u = g(u, \nabla u)$$

Convolution

$f(x), g(x)$ defined on $\mathbb{R}$.

$f \in L^1(\mathbb{R}) \qquad g \in L^p(\mathbb{R}) \qquad 1 \leq p \leq \infty$

We define a new function

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy$$

convolution of f and g.

THEOREM $f \in L^1(\mathbb{R}),\ g \in L^p(\mathbb{R}) \qquad 1 \leq p \leq \infty$

$\Rightarrow$ 1) $(f * g)(x)$ is well defined for a.e. $x \in \mathbb{R}$.

2) $f * g \in L^p(\mathbb{R})$

3) $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

Note $\quad f * g = g * f$, indeed.

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy$$

change of variable

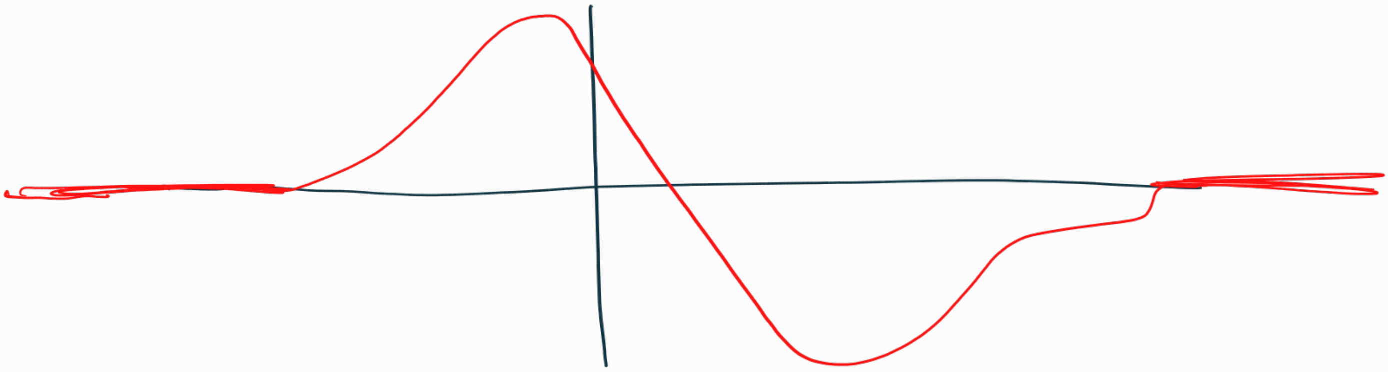
$x - y = \tilde{y}$

$y = x - \tilde{y}$

$$= \int_{\mathbb{R}} f(\tilde{y})\, g(x-\tilde{y})\, d\tilde{y} = (g * f)(x)$$

Regularity of convolutions

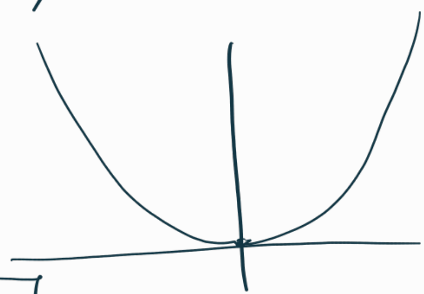
$f \in C_c(\mathbb{R}) = \{$ continuous functions on $\mathbb{R}$ with compact support, i.e. $f \equiv 0$ outside of a bounded interval $[a,b]$ $\}$



$g \in L^1_{loc}(\mathbb{R})$ i.e. $g : \mathbb{R} \to \mathbb{R}$ s.t. $g \in L^1(a,b)$ $\forall -\infty < a < b < +\infty$

For instance $g(x) = x^2 \notin L^1(\mathbb{R})$

$x^2 \in L^1_{loc}(\mathbb{R})$



THEOREM $f \in C_c(\mathbb{R})$, $g \in L^1_{loc}(\mathbb{R})$

$\Rightarrow$ $f * g \in C(\mathbb{R})$

Remark $f * g$ is well defined

$$(f*g)(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy.$$

$\in L^1$ on the interval where $f(x-y) \neq 0$

continuous $\Rightarrow$ bounded and different from zero on a bdd interval

Proof I want to show that $f*g$ is continuous, i.e. if $x_n \to x \in \mathbb{R}$ $(f*g)(x_n) \overset{?}{\to} (f*g)(x)$

$$\int_{\mathbb{R}} \underbrace{f(x_n-y)\, g(y)}_{F_n(y)}\, dy \xrightarrow{n\to+\infty} \int_{\mathbb{R}} \underbrace{f(x-y)\, g(y)}_{F(y)}\, dy$$

1) $F_n(y) \to F(y)$. a.e in $\mathbb{R}$

2) $|F_n(y)| = \underbrace{|f(x_n-y)|}_{}\, |g(y)| \leq \|f\|_\infty \overbrace{\chi_I\, |g(y)|}^{\in L^1} \in L^1(\mathbb{R})$

zero outside of a bounded interval I indipendent on n,
Moreover it is bounded by some K.

$\Rightarrow$ Lebesgue's dominated convergence theorem

$$\int F_n(y) \to \int F(y)$$ $\square$

What about derivatives?

THEOREM

$f \in C^1_c(\mathbb{R})$ ($C^1(\mathbb{R})$ and zero outside of a bounded interval.)

$g \in L^1_{loc}(\mathbb{R})$

$\Rightarrow f*g \in C^1(\mathbb{R})$

and $(f*g)' = f' * g$

Proof

$$(f*g)'(x) = \lim_{h\to 0} \frac{(f*g)(x+h)-(f*g)(x)}{h} =$$

$$= \lim_{h\to 0} \frac{\int_{\mathbb{R}} f(x+h-y)\,g(y)\,dy - \int_{\mathbb{R}} f(x-y)\,g(y)\,dy}{h}$$

$$= \lim_{h\to 0} \int_{\mathbb{R}} \frac{f(x+h-y)-f(x-y)}{h}\, g(y)\, dy =$$

by Lagrange's theorem

$$\frac{f(x+h-y)-f(x-y)}{h} = f'(x-y+t_h h)$$

with $t_h \in (0,1)$

$$= \lim_{h\to 0} \int_{\mathbb{R}} f'(\underbrace{x-y+t_h h}_{(x+t_h h-y)})\, g(y)\, dy =$$

$$= \lim_{h\to 0} (f'*g)(x+t_h h) = (f'*g)(x) \qquad \square$$

Note $f' \in C_c(\mathbb{R}) \Rightarrow f'*g$ is continuous by the previous thm

Note of course if $f \in C_c^2(\mathbb{R}) \Rightarrow (f*g) \in C^2(\mathbb{R})$

and $(f*g)'' = f''*g$, and so on.

These definitions and results work also in dimension $N>1$.

$f \in L^p(\mathbb{R}^N)$, $g \in L^1(\mathbb{R}^N)$

$$\Rightarrow (f*g)(\underline{x}) = \int_{\mathbb{R}^N} f(\underline{x}-\underline{y})\, g(\underline{y})\, d\underline{y} =$$

$$= \int_{\mathbb{R}} dy_1 \int_{\mathbb{R}} dy_2 \cdots \int_{\mathbb{R}} dy_N\, f(x_1-y_1, x_2-y_2, \dots, x_N-y_N)\, g(y_1, y_2, \dots y_N)$$

and if $f \in C^1_c(\mathbb{R}^N)$, $g \in L^1_{loc}(\mathbb{R}^N)$

$$\Rightarrow f*g \in C^1(\mathbb{R}^N)$$

$$\nabla(f*g) = (\nabla f)*g$$

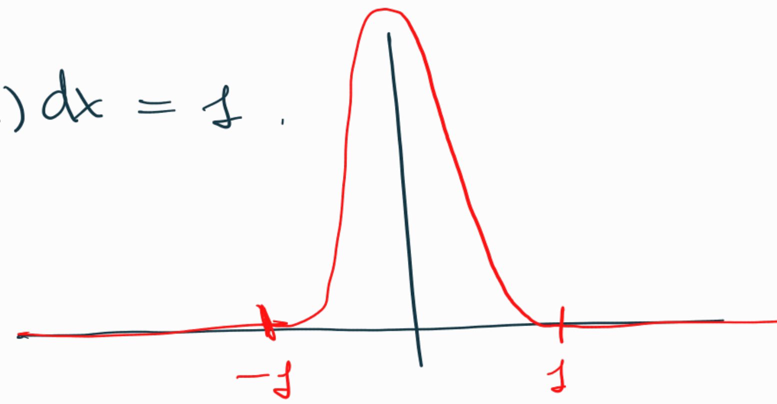
<u>Mollifiers.</u>

Let $\rho(x) \in C^\infty_c(\mathbb{R})$ (differentiable any number of times at any point, and zero outside of a bounded interval.)

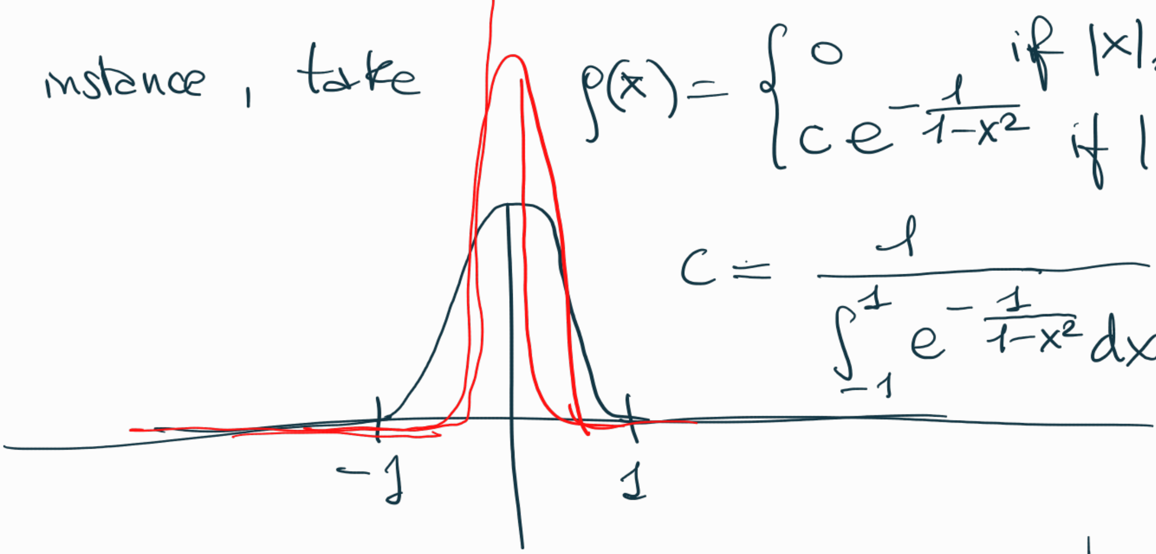
$$\text{supp}\, \rho(x) \subset [-1,1] \Rightarrow \rho(x) = 0 \text{ outside of } [-1,1]$$

$$\rho(x) \geq 0$$

$$\int_{\mathbb{R}} \rho(x)dx = \int_{-1}^{1} \rho(x)\, dx = 1.$$



For instance, take



$$\rho(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \\ c\, e^{-\frac{1}{1-x^2}} & \text{if } |x| < 1. \end{cases}$$

$$c = \frac{1}{\int_{-1}^{1} e^{-\frac{1}{1-x^2}}\,dx}$$

We define the mollifiers.

$$\rho_n(x) = n\,\rho(nx)$$

$$\int_{\mathbb{R}} \rho_n(x)\,dx = n \int_{\mathbb{R}} \rho(nx)\,dx =$$

$nx = t$

$dx = \frac{dt}{n}$

$$= \frac{\cancel{n}}{\cancel{n}} \int_{\mathbb{R}} \rho(t)\,dt = 1$$

"$\rho_n(x)$ tend to a Dirac δ at $x=0$".

THEOREM.

Let $f \in L^p(\mathbb{R}) \subset L^p_{loc}(\mathbb{R}) \subset L^1_{loc}(\mathbb{R})$ $\quad 1 \leq p < \infty$

let $\rho_n(x)$ be a sequence of mollifiers.

Then $\underbrace{\rho_n * f}_{\in\, C^\infty(\mathbb{R})} \xrightarrow{n\to\infty} f$ in $L^p(\mathbb{R})$.

Remark. This cannot be true in $L^\infty(\mathbb{R})$

Indeed

$$f * \rho_n \in C^\infty(\mathbb{R}) \subset C(\mathbb{R})$$

If $f * \rho_n \longrightarrow f$ in L^∞, then it converges uniformly, so f must be continuous.

The proof is made of two subsequent approximations:

1) Approximation with functions which are continuous with compact support.

<u>THEOREM</u> $1 \leq p < \infty$, Ω open set in $\mathbb{R}^N$,
Then $\forall f \in L^p(\Omega)$, $\forall \varepsilon > 0$
$\exists\ g_\varepsilon \in C_c(\Omega)$ s.t. $\|f - g_\varepsilon\|_p < \varepsilon$.

($C_c(\Omega)$ is dense in $L^p(\Omega)$).

<u>Idea of the proof</u> for $p = 1$, and $\Omega = \mathbb{R}$. $(N=1)$

$$f \in L^p(\mathbb{R})$$

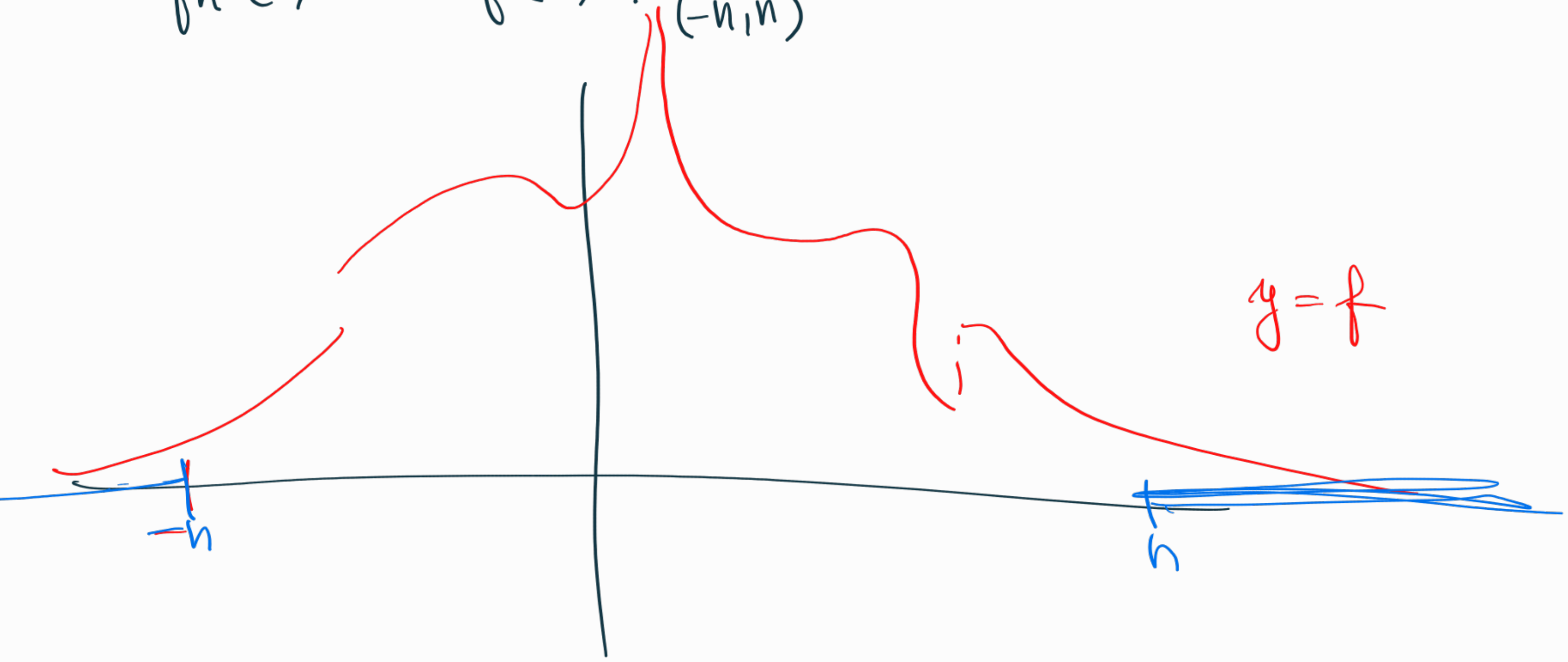
<u>Step 1</u> I can write $f = f^+ - f^-$ $\qquad f^+(x) = \max\{0, f(x)\}$

and approximate each of them with $\qquad f^-(x) = (-f)^+(x)$

a $g_\varepsilon \Rightarrow$ I can take $f \geq 0$.

Step 2. Reduction to compact support.

$$f_n(x) = f(x)\,\chi_{(-n,n)}(x)$$



$$\left.\begin{array}{l} f_n(x) \longrightarrow f(x) \quad a.e. \\ 0 \le f_n(x) \le f(x) \end{array}\right| \Rightarrow$$

$$\left.\begin{array}{l} |f(x)-f_n(x)| = f(x)-f_n(x) \longrightarrow 0 \\ \text{and } |f(x)-f_n(x)| \le f(x) \in L^1(\mathbb{R}) \end{array}\right) \text{Lebesgue dominated converge}$$

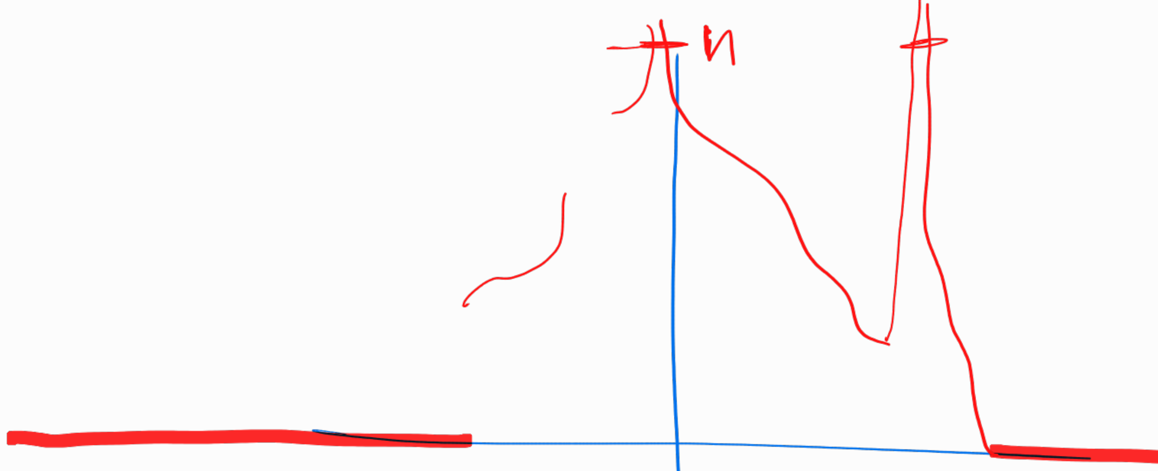
$$\Rightarrow \int_{\mathbb{R}} |f(x)-f_n(x)|\,dx \to 0 \Rightarrow f_n \to f \quad L^1\,\mathbb{R}.$$

So if n is large enough $\|f_n - f\|_1 < \varepsilon$

Step 3 Reduction to bounded functions

Assume that $f \equiv 0$ outside of a compact set

$$f_n(x) = T_n(f(x)) = \begin{cases} n & \text{if } f(x) > n \\ f(x) & 0 \le f(x) \le n. \end{cases}$$



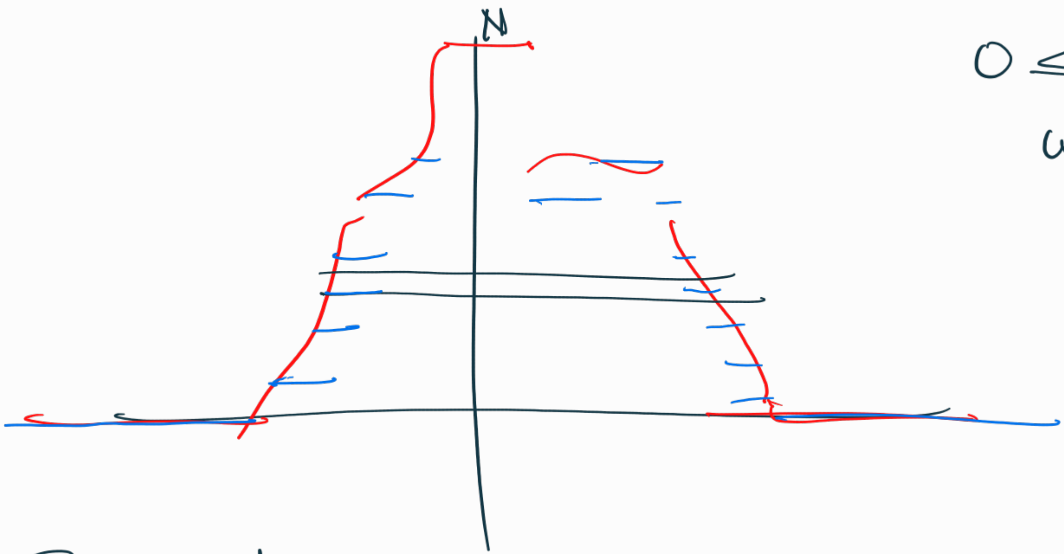
$f_n(x) \to f(x)$ a.e.

$|f_n(x)| \le |f(x)|$

$\Rightarrow f_n(x) \longrightarrow f$ a.e

Therefore we can take n such that $\|f - f_n\|_1 < \varepsilon$

Step 4. Approximation with simple functions.



$0 \le f(x) \le N$

with compact support

Fix $n \in \mathbb{N}$.

$$E_{i,n} = \left\{ x : f(x) \in \left[\frac{i-1}{n}, \frac{i}{n}\right) \right\}$$

$$i = 1, \dots, (n+1)N.$$

$\varphi_n(x) = \dfrac{i-1}{n}$ on the set $E_{i,n}$

$$\varphi_n(x) = \sum_{i=1}^{(n+1)N} \frac{i-1}{n} \chi_{E_{i,n}}(x)$$

It can be checked that

$0\le \varphi_n(x)\le f$
$\varphi_n(x)\to f(x)$ a.e. $\Big|\Rightarrow \varphi_n \to f \quad L^1(\Omega)$

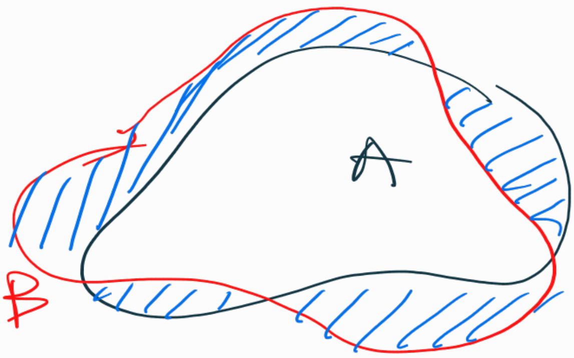
Therefore we can take n large enough so that $\|f-\varphi_n\|<\varepsilon$.

Step 5 We approximate the sets $E_{i,n}$ with finite unions of intervals.

LEMMA If $E\subset\mathbb{R}$ is measurable and has finite measure $\forall\varepsilon>0$ one can find a finite number of open intervals I_j, $j=1\dots m$ s.t.

$$m\left(E\,\triangle\left(\bigcup_{j=1}^{m} I_j\right)\right)<\varepsilon.$$

$$A\triangle B=(A\setminus B)\cup(B\setminus A)$$



At the end of this process, I can approximate f with a function of the form

$$\sum_{j=1}^{M}\lambda_j\chi_{E_j}$$ where E_j are disjoint intervals.

Step 6 Now we approximate the step function which we have obtained at step 5 with continuous function, by joining the steps with steep ramps.

Now we can assume that $f \in C_c(\mathbb{R})$ and I want to approximate it with functions in $C_c^\infty(\mathbb{R})$.

I use the following theorem:

$$\boxed{\begin{array}{l}\underline{\text{Prop}} \text{ (Heine-Cantor theorem).} \\ f \text{ continuous in } [a,b] \Rightarrow f \text{ is uniformly continuous in } [a,b]\end{array}}$$

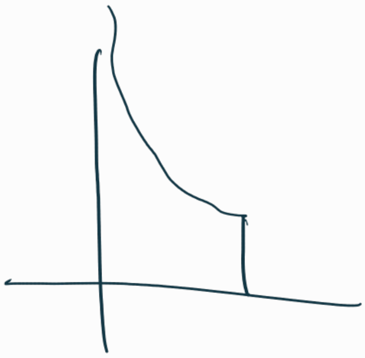
f is uniformly continuous if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \forall x, y \in [a,b] \text{ s.t. } |x-y| < \delta$$
$$\Rightarrow |f(x) - f(y)| < \varepsilon.$$

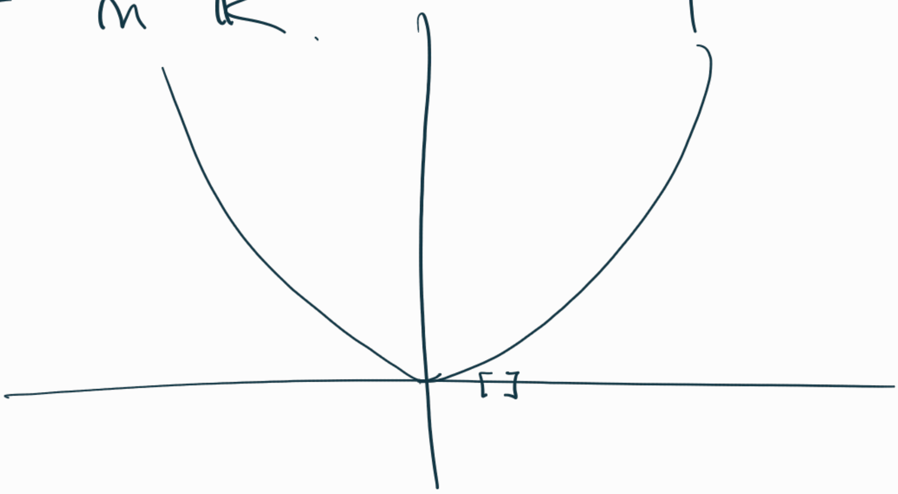
Note δ depends on ε, but not on x

Example of a continuous function which is not uniformly continuous

$f(x) = \frac{1}{x}$ in $(0,1]$



$f(x) = x^2$ in $\mathbb{R}$.



I take $\rho_n(x)$ to be a suite of mollifiers.
and I want to prove that.

$$(\rho_n * f)(x) \xrightarrow{n} f(x) \qquad \forall x \in \mathbb{R}.$$

$$\left|(\rho_n * f)(x) - f(x)\right| = \left|\int_{\mathbb{R}} f(x-y)\rho_n(y)\,dy - \Big(\underbrace{\int_{\mathbb{R}} \rho_n(y)dy}_{=1}\Big) f(x)\right| =$$

$$= \Big|\int_{\mathbb{R}} [f(x-y) - f(x)]\, \underbrace{\rho_n(y)}_{\text{has support in } [-\frac{1}{n}, \frac{1}{n}]}\, dy\Big| =$$

$$= \Big|\int_{[-\frac{1}{n}, \frac{1}{n}]} \big(f(x-y) - f(x)\big)\, \rho_n(y)\, dy\Big| \leq \text{triangle ineq}$$

$$\leq \int_{[-\frac{1}{n}, \frac{1}{n}]} |f(x-y) - f(x)|\, \rho_n(y)\, dy \leq$$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

$\Rightarrow$ I fix $n > \frac{1}{\delta} \Rightarrow \frac{1}{n} < \delta \Longrightarrow |(x-y) - x| \,(= |y|) \leq \frac{1}{n} < \delta$

$\Longrightarrow |f(x-y) - f(x)| < \varepsilon$

$$< \varepsilon \underbrace{\int_{[-\frac{1}{n}, \frac{1}{n}]} \rho_n(y)\,dy}_{1} = \varepsilon$$