<u>DEF.</u> Sobolev Space $W^{1,p}(\Omega)$ in dim $N \geq 2$.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded, open set in $\mathbb{R}^N$.

$\boxed{1 \leq p \leq \infty}$

$W^{1,p}(\Omega) = \{ u \in L^p(\Omega)$ which admits weak partial derivatives $V_1, V_2, \dots V_N \in L^p(\Omega)$ s.t.

$$\int_\Omega u \, \varphi_{x_i} \, dx = - \int_\Omega V_i \, \varphi \, dx \quad \forall \varphi \in C_c^1(\Omega) \}$$

as before, we will denote the weak derivatives V_i by $\dfrac{\partial u}{\partial x_i} = u_{x_i}$

and $(V_1, V_2, \dots V_N)$ by ∇u.

This is a vector space, and it is a normed space with the norm.

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$$

$$= \|u\|_{L^p(\Omega)} + \left[\int_\Omega |\nabla u|^p \, dx\right]^{1/p}$$

or we could take

$$\|u\|_{W^{1,p}(\Omega)} = \left[\int_\Omega \left(|u|^p + \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^p\right) dx\right]^{1/p}$$

These norms are equivalent. The space $W^{1,p}(\Omega)$ is a Banach space

In the special case $p=2$ $W^{1,p}(\Omega)$ is a Hilbert space with the scalar product.

$$(u,v)_{W^{1,2}(\Omega)} = \int_\Omega u v \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx =$$

$$= (u,v)_{L^2(\Omega)} + \sum_{i=1}^N (u_{x_i}, v_{x_i})_{L^2(\Omega)}$$

$$W^{1,2}(\Omega) = H^1(\Omega)$$

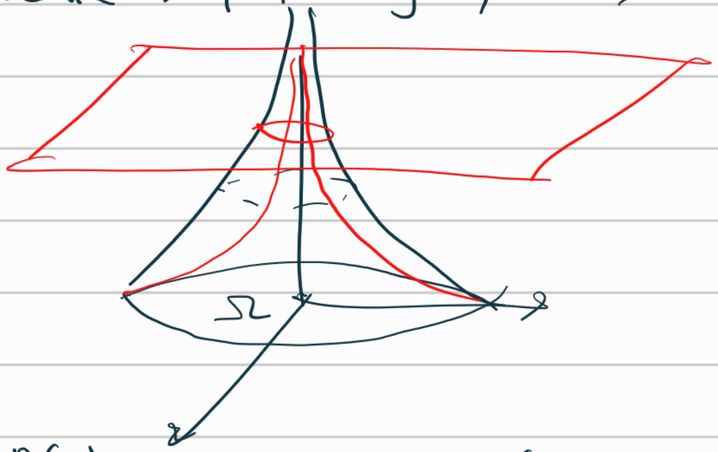
Examples of functions in $W^{1,p}(\Omega)$

If $u \in C^1(\overline{\Omega}) \Rightarrow u \in W^{1,p}(\Omega)$ $\forall p \in [1, +\infty]$

Remark. Since $m(\Omega) < \infty$, if $1 \leq p < q \leq \infty$, then $W^{1,q}(\Omega) \subset W^{1,p}(\Omega)$ and $\|u\|_{W^{1,p}} \leq C(\Omega) \|u\|_{W^{1,q}}$

Example $\Omega = B_1(0) = \{ x \in \mathbb{R}^N : |x| < 1 \}$, $N \geq 2$

$$u(x) = -\ln|x| = \ln \frac{1}{|x|}$$



I claim that $u(x) \in W^{1,p}(\Omega)$ for some p (to be computed)

$u \overset{?}{\in} L^p(\Omega)$ $\forall\, p \in [1, +\infty)$

I also claim that the weak derivatives of u are

$$(*) \quad \boxed{u_{x_i}(x) = -\frac{1}{|x|}\,(|x|)_{x_i} = -\frac{x_i}{|x|^2}}$$

(i.e. it coincides with the "classical" derivative outside the origine.)

$$\frac{\partial}{\partial x_i}|x| = \frac{\partial}{\partial x_i}\sqrt{\sum_{j=1}^{N} x_j^2} = \frac{\cancel{2}x_i}{\cancel{2}\sqrt{\sum_j x_j^2}} = \frac{x_i}{|x|}$$

$u_{x_i}(x) \in L^p(\Omega)$ for some $p \geq 1$?

$$u_{x_i}(x) = -\frac{x_i}{|x|^2}$$

$$|u_{x_i}(x)| = \frac{|x_i|}{|x|^2} \leq \frac{1}{|x|}$$

$\Leftarrow |x|$

$$\Rightarrow \int_\Omega |u_{x_i}(x)|^p \leq \int_\Omega \frac{1}{|x|^p} < \infty \iff \boxed{p < N}$$

So $u \in W^{1,p}(\Omega) \quad \forall p < N$

So we only have to prove that the weak derivative

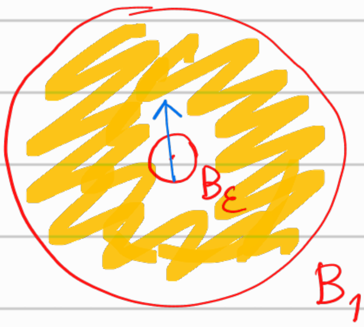
u_{x_i} is $-\dfrac{x_i}{|x|^2}$

This means:

$$\int_\Omega u\,\varphi_{x_i}\,dx = -\int_\Omega u_{x_i}\,\varphi\,dx \qquad \forall \varphi \in C^1_c(\Omega)$$

$$-\int_{\underset{\underset{B_1}{\parallel}}{\Omega}} \log|x|\,\varphi_{x_i}\,dx \overset{?}{=} +\int_\Omega \frac{x_i}{|x|^2}\,\varphi\,dx$$

$$\parallel$$



$$\underbrace{-\int_{B_1\setminus B_\varepsilon} \log|x|\,\varphi_{x_i}\,dx}_{\text{Here the functions are regular: I can integrate by parts.}} - \underbrace{\int_{B_\varepsilon} \log|x|\,\varphi_{x_i}\,dx}_{=\,o(1) \text{ for } \varepsilon\to 0}$$

$$\parallel$$

$$\int_{B_1\setminus B_\varepsilon} \frac{x_i}{|x|^2}\,\varphi\,dx + \int_{\partial B_1} \underbrace{u\,\varphi\,\nu_i}_{=\,0\ (\varphi=0 \text{ on } \partial B_1)}\,d\sigma + \underbrace{\int_{\partial B_\varepsilon} \overset{-\log\varepsilon}{\overset{\parallel}{u}}\,\varphi\,\nu_i\,d\sigma}_{\parallel}$$

$$\underbrace{-\int_{\partial B_\varepsilon} \log\varepsilon\,\underset{\underset{C}{\wedge\!\shortparallel}}{\varphi\,\nu_i}\,d\sigma}$$

$$= \int_{B_1 \setminus B_\varepsilon} \frac{x_i}{|x|^2} \varphi \, dx + o(1)$$

In absolut value this integral is bounded by

$$\underbrace{-c \log \varepsilon \, \underbrace{\text{area}(\partial B_\varepsilon)}_{= c\varepsilon^{N-1}}}_{\to 0 \text{ as } \varepsilon \to 0}$$

$$= \int_{B_1} \frac{x_i}{|x|^2} \varphi \, dx - \underbrace{\int_{B_\varepsilon} \frac{x_i}{|x|^2} \varphi \, dx}_{= o(1)} + o(1)$$

Sobolev functions can be pretty irregular:

for instance

$$u(x) = \sum_{k=0}^{\infty} \frac{(-\ln|x - x_k|)}{2^k} \in L^p(B_1) \ \forall p < N$$

x_k sequence of all points with rational coordinates in B_1

So, in general, functions in Sobolev spaces $(N > 1)$ need not be continuous.

But anyway, they are "more regular" than an L^p function

Theorem.

Let $\Omega \subset \mathbb{R}^N$ bounded open set, let $u \in W^{1,p}(\Omega)$

Then.

1) If $p < N$, then $u \in L^{p^*}(\Omega)$, where

$$p^* = \frac{pN}{N-p} > p \quad (\Rightarrow u \in L^q(\Omega) \ \forall q \le p^*)$$

(Sobolev)

2) If $p=N$, then $u \in L^q(\Omega)$ $\forall q<\infty$, and

$$\int_\Omega e^{\alpha|u|}dx < \infty \quad \text{for some } \alpha>0$$ (Trudinger)

3) If $p>N$, then $u \in C(\overline{\Omega})$ (Morrey)

I want to define

$W_0^{1,p}(\Omega)$ = closure of $C_c^1(\Omega)$ in the $W^{1,p}(\Omega)$ norm =

$C_c^\infty(\Omega)$

$= \{ u \in W^{1,p}(\Omega) \text{ s.t. } \exists \{\varphi_n\} \subset C_c^1(\Omega) \text{ s.t. }$
$\varphi_n \longrightarrow u \text{ in } W^{1,p}(\Omega) \}$

$= \{ u \in W^{1,p}(\Omega)$ which "vanish on $\partial\Omega$" $\}$

I use quotes because u is an L^p function, so they are not defined on sets of zero measures, such as $\partial\Omega$.
In order to explain the sentence "which vanish on $\partial\Omega$", I need to talk about traces of $W^{1,p}(\Omega)$ functions.

Trace theorem

Ω bounded open set in $\mathbb{R}^N$, with smooth boundary
$1 \le p \le \infty$.

Then $\exists$ T linear operator

$$T: W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$$

s.t.

i) if $u \in W^{1,p} \cap C(\overline{\Omega}) \Rightarrow Tu = u|_{\partial\Omega}$

ii) $\|Tu\|_{L^p(\partial\Omega)} \le c \, \|u\|_{W^{1,p}(\Omega)} \quad \forall u \in W^{1,p}(\Omega)$

$c = c(p, N, \Omega)$

Tu is called the trace of u on $\partial\Omega$.

$W_0^{1,p}(\Omega)$ is the set of functions in $W^{1,p}(\Omega)$ which have 0 trace on $\partial\Omega$.

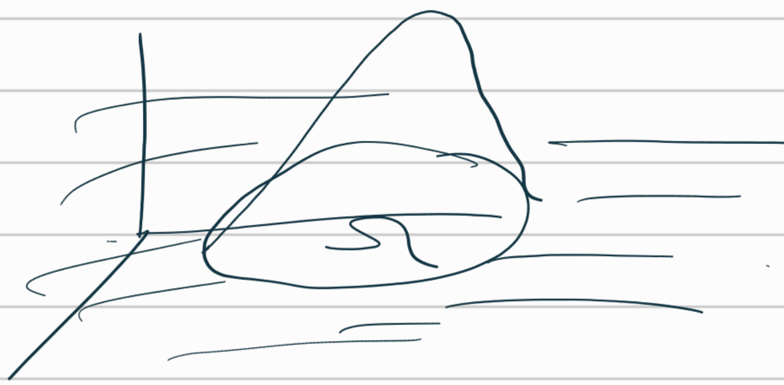
THEOREM (Characterization of $W_0^{1,p}(\Omega)$).

Same hypotheses as before, then the following statements are equivalent.

1) $u \in W_0^{1,p}(\Omega)$

2) $Tu = 0$

3) $\tilde{u}(x) = \begin{cases} u(x) & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \in W^{1,p}(\mathbb{R}^N)$



Dirichlet problem for Poisson's equation

$$(P)\begin{cases} \underbrace{-\Delta u}_{-\sum_{i=1}^{N} u_{x_i x_i}} + u = \underbrace{f(x)}_{\text{given}} & \text{in } \Omega \subset \mathbb{R}^N \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Classical solution : $u \in C^2(\overline{\Omega})$ which satisfies the first equality at all points of Ω, and s.t. $u = 0$ on $\partial\Omega$.

Our plan is in four steps, as in dim. 1.

1) We define a weak solution of (P) in such a way that a classical solution is also a weak solution

2) Functional Analysis $\Rightarrow$ $\exists!$ weak solution of (P) under weak assumptions on f $\quad (f \in L^2(\Omega))$

3) Regularity: If f is "more regular", then a weak solution is "more regular", hopefully $C^2(\overline{\Omega})$.

4) If u is a weak sol. of (P) of class $C^2(\overline{\Omega})$, then u is a classical solution

1) def of weak solution.

If $u \in C^2(\overline{\Omega})$ is a classical solution, then

$$\underbrace{\int_\Omega (-\Delta u)\varphi \, dx}_{4} + \int_\Omega u\varphi \, dx = \int_\Omega f\varphi \, dx \quad \forall \varphi \in C_c^1(\Omega)$$

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx$$ + boundary term which is zero because $\varphi = 0$ on $\partial\Omega$

$$\Rightarrow \int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Omega u \varphi \, dx = \int_\Omega f \varphi \, dx \quad \forall \varphi \in C_c^1(\Omega)$$

So, a <u>weak solution</u> of problem (P) is a function $u \in W_0^{1,2}(\Omega) = H_0^1(\Omega)$ s.t.

$$\int_\Omega \nabla u \cdot \nabla \varphi_n \, dx + \int_\Omega u \varphi_n \, dx = \int_\Omega f \varphi_n \, dx \quad \forall \varphi \in C_c^1(\Omega)$$

If $v \in H_0^1(\Omega) \Rightarrow \exists \varphi_n \in C_c^1(\Omega)$ s.t. $\varphi_n \to v \quad H_0^1(\Omega)$

$$\Rightarrow \underbrace{\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u v \, dx}_{(u,v)_{H_0^1}} = \int_\Omega f v \, dx \quad \forall v \in H_0^1(\Omega)$$

Let us prove <u>Step 2</u> (Existence and uniqueness).

<u>Theorem</u> $\forall f \in L^2(\Omega) \; \exists! \; u \in H_0^1(\Omega)$ weak sol. of (P)

<u>Proof.</u> We need to show that $\forall f \in L^2(\Omega)$ $\exists! \; u \in H_0^1(\Omega)$ s.t.

$$(u,v)_{H_0^1(\Omega)} = \overbrace{\int_\Omega f v \, dx}^{= L v} \quad \forall v \in H_0^1(\Omega)$$

<u>Theorem (Riesz-Frechet)</u> H Hilbert space, $L : H \to \mathbb{R}$ linear and continuous ($|L v| \le M \|v\|_H$)
Then $\exists! \; \xi \in H$ s.t.

$$L v = (\xi, v) \quad \forall v \in H$$

I define $\quad Lv = \int_\Omega f v \, dx = (f,v)_{L^2(\Omega)}$

$Lv : H_0^1(\Omega) \to \mathbb{R}$ $\quad$ linear

$$|Lv| = (f,v)_{L^2(\Omega)} \leq \|f\|_{L^2} \underbrace{\|v\|_{L^2}}_{\substack{\wedge\\ \|v\|_{H_0^1}}} \leq \|f\|_{L^2} \|v\|_{H_0^1}$$

$\Rightarrow$ R-F theorem $\Rightarrow \exists! \, u \in H_0^1(\Omega)$ s.t.

$$(u,v)_{H_0^1(\Omega)} = Lv = \int_\Omega f v \, dx \qquad \forall v \in H_0^1(\Omega)$$

Step 3 is not easy (but true).

Step 4 If u is a weak solution, $\underbrace{u \in H_0^1(\Omega) \cap C^2(\overline{\Omega})}$,

then u is a classical solution. $\quad \Rightarrow u(x) = 0$ on $\partial\Omega$

$$\underbrace{\int_\Omega \nabla u \cdot \nabla \varphi \, dx}_{\substack{\| \\ \int_\Omega -\Delta u \, \varphi \, dx}} + \int_\Omega u \varphi \, dx = \int_\Omega f \varphi \, dx \quad \forall \varphi \in C_c^1(\Omega)$$

$$\int_\Omega (-\Delta u + u - f) \varphi \, dx = 0 \qquad \forall \varphi \in C_c^1(\Omega)$$

We use the fund. lemma of calculus of Variations.

$$\Rightarrow \underbrace{(-\Delta u + u - f)}_{\text{continuous}} = 0 \quad \text{a.e. in } \Omega.$$

$$\Rightarrow \quad -\Delta u + u = f \qquad \forall x \in \Omega.$$

Extensions.

no term "$+u$"

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

1) Weak sol^n.

$u \in H_0^1(\Omega)$ s.t. $\displaystyle\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \quad \forall \varphi \in \cancel{C_c^1(\Omega)}$ $\in H_0^1(\Omega)$

2) Here we cannot proceed as before.

3)
4) as before

$\boxed{\text{We use the following Poincaré's inequality.}}$

We use the following Poincaré's inequality.
$\Omega \subset \mathbb{R}^N$ bounded open set, $\exists\, c = c(\Omega)$ s.t.

$$\int_\Omega u^2 \, dx \leq c \int_\Omega |\nabla u|^2 \, dx \qquad \forall u \in H_0^1(\Omega)$$

Consequence:
In $H_0^1(\Omega)$ instead of taking the norm

$$\|u\|^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$$

we can take.

$$|||u|||^2 = \|\nabla u\|_{L^2}^2$$

$$c\,\|u\| \le |||u||| \le \|u\|$$

So, these are equivalent norms on $H^1_0(\Omega)$.

The new norm $|||u|||$ corresponds to a scalar product

$$(u,v) = \int_\Omega \nabla u\cdot\nabla v\,dx$$

So, to prove existence and uniqueness we have just to apply Riesz-Frechet theorem with this (equivalent) scalar product.

$$\int_\Omega \nabla u\cdot\nabla v\,dx = \underbrace{\int_\Omega f\,v\,dx}_{} \qquad \forall v\in H^1_0(\Omega)$$

$$(u,v) = Lv$$

$$|Lv| = \left|\int_\Omega f\,v\,dx\right| \le \|f\|_{L^2}\,\|v\|_{L^2} \le \|f\|_{L^2}\,c\,\|\nabla v\|_{L^2}$$
$$= c\,\|f\|_{L^2}\,|||v|||$$

Poisson equation with Neumann condition

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

Def of weak solution

$u \in H^1(\Omega) = W^{1,2}(\Omega)$ (no value prescribed on $\partial\Omega$)
s.t.

$$(*) \quad \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u v \, dx = \int_\Omega f v \, dx \quad \forall v \in H^1(\Omega)$$

Let us check that a classical solution is also a weak sol.

$u \in C^2(\overline{\Omega})$, $\quad -\Delta u + u = f \quad$ pointwise

$$\underbrace{-\int_\Omega \Delta u \, \varphi \, dx}_{\displaystyle \parallel} + \int_\Omega u \varphi \, dx = \int_\Omega f \varphi \, dx \qquad \forall \varphi \in C^1(\overline{\Omega})$$

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial \nu}}_{\overset{\parallel}{0}} \varphi \, dx$$

and then, approximating $v \in H^1(\Omega)$ with $\varphi_n \in C^1(\overline{\Omega})$, we obtain $(*)$

2) $\exists !$ of a weak sol^n $\to$ similar to previous case

3) Regularity (difficult but true)

4) How do we prove that a smooth weak solution is a classical solution

$u \in H^1(\Omega) \cap C^2(\overline{\Omega})$ s.t.

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \, dx}_{} + \int_\Omega u v \, dx = \int_\Omega f v \, dx \qquad \forall v \in C^1(\overline{\Omega})$$

$$\parallel$$

$$-\int \Delta u \, v \, dx$$

I can take $v \in C_c^1(\Omega)$

$$\Rightarrow \int_\Omega (-\Delta u + u - f) \, v \, dx = 0 \quad \forall v \in C_c^1(\Omega)$$

$$\Rightarrow \underbrace{-\Delta u + u - f}_{\text{continuous}} = 0 \quad \text{a.e}$$

$$\Rightarrow -\Delta u + u = f \quad \text{at every point.}$$

Now we take $\varphi \in C^1(\overline{\Omega})$

$$\underbrace{\int_\Omega \nabla u \cdot \nabla \varphi \, dx}_{\parallel} + \cancel{\int_\Omega u \varphi \, dx} = \cancel{\int_\Omega f \varphi \, dx}$$

$$\cancel{\int_\Omega (-\Delta u) \varphi \, dx} + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \varphi \, d\sigma$$

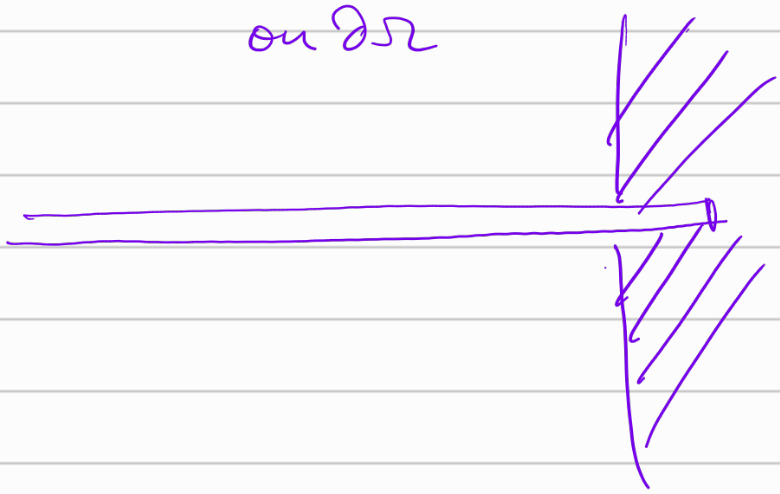
$$\Rightarrow \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \varphi \, d\sigma = 0 \quad \forall \varphi \in C^1(\overline{\Omega})$$

$$\Downarrow$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

Problem of "clamped" plate

$$(P)\quad\begin{cases} \Delta^2 u = f(x) \text{ given} \quad \text{on } \Omega \subset \mathbb{R}^2 \\ \Delta(\Delta u) = \Delta(u_{xx}+u_{yy}) = u_{xxxx} + 2u_{xxyy} + u_{yyyy} \\ u = 0 \quad \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{cases}$$



Classical solution of (P)

$u \in C^4(\overline{\Omega})$ s.t. the equations are satisfied pointwise

Weak solution.

$u \in H^2_0(\Omega) = \underbrace{W^{2,2}}_{\text{Sobolev space of function with weak second derivatives in } L^2(\Omega)}_{0}(\Omega)$

W_0: u is the limit in $W^{2,2}(\Omega)$ of functions $C^2_c(\Omega)$

$C^2_c(\Omega)$ = twice differentiable functions with continuous derivatives and zero near the boundary

The weak formulation becomes

$$\int_\Omega \Delta u \, \Delta v \, dx = \int_\Omega f v \, dx \quad \forall v \in H_0^2(\Omega)$$

In a similar way one can deal with equations of the form

$$-\operatorname{div}(A(x)\nabla u) = f \qquad + \text{b.c.}$$

$$-\operatorname{div}(A(\nabla u)) = f$$

under reasonable assumptions on the function A.

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \qquad p \neq 2$$

p-Laplacian. $\qquad p > 1$