LEMMA

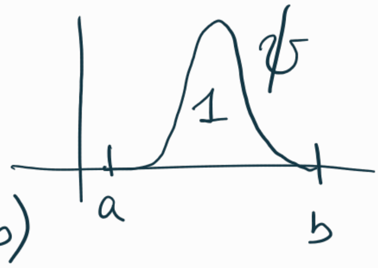
Assume that $f \in L^1(a,b)$ s.t. $\int_a^b f \varphi' \, dx = 0 \quad \forall \varphi \in C_c^1(a,b)$. $(*)$

(i.e., $f \in W^{1,1}(a,b)$, $f' = 0$)

$\Rightarrow \exists c \in \mathbb{R}$ s.t. $f(x) = c$ a.e. on (a,b).

<u>Proof</u> Let us fix a $\psi \in C_c^1(a,b)$ s.t. $\int_a^b \psi(x)\,dx = 1$.

Let us take any function $w \in C_c^1(a,b)$.



I claim that there exists a function $\varphi \in C_c^1(a,b)$

s.t. $\boxed{\varphi'(x) = w(x) - \left(\int_a^b w\right) \psi(x)}$ (proof later) (1)

Let us take φ in $(*)$

$$\Rightarrow \int_a^b f(x) \left[w(x) - \left(\int_a^b w(t)\,dt \right) \psi(x) \right] dx = 0$$

$$\Rightarrow \underbrace{\int_a^b f(x)\, w(x)\,dx}_{= \int_a^b f(t)\, w(t)\,dt} - \underbrace{\int_a^b f(x)\, \psi(x) \left(\int_a^b w(t)\,dt \right) dx}_{= \int_a^b w(t)\,dt \int_a^b f(x)\, \psi(x)\,dx} = 0$$

$$\Rightarrow \int_a^b w(t) \Big[\underbrace{f(t) - \int_a^b f(x)\, \psi(x)\,dx}_{g(t) \in L^1(a,b)} \Big] dt = 0$$

$$\Rightarrow \int_a^b w(t)\, g(t)\, dt = 0 \qquad \forall w \in C_c^1(a,b)$$

$\Rightarrow$ Fund. lemma of the calculus of variation

$$\Rightarrow g(t) \equiv 0 \quad \text{a.e. in } (a,b)$$

$$\Rightarrow f(t) = \underbrace{\int_a^b f(x)\psi(x)\,dx}_{C} \quad \text{a.e. } t \in (a,b)$$

So we only have to show that there exist $\varphi \in C_c^1(a,b)$ s.t.

$$\varphi'(x) = w(x) - \left(\int_a^b w\right)\psi(x)$$

we define

$$\varphi(x) = \int_a^x \left[w(t) - \left(\int_a^b w\right)\psi(t)\right] dt.$$

$$\Rightarrow \varphi'(x) = w(x) - \left(\int_a^b w\right)\psi(x)$$ by fund. theorem of Integral Calculus.

$\varphi(a) = 0$ and this is also true in a right neighbd of a

$$\varphi(b) = \int_a^b \left[w(t) - \left(\int_a^b w(x)dx\right)\psi(t)\right] dt =$$

$$= \int_a^b w(t)\,dt - \int_a^b w(x)\,dx \underbrace{\boxed{\int_a^b \psi(t)\,dt}}_{=1} = 0$$

and this is also true in a left ngbh. of b. $\square$

Linear continuous operators

X, Y are normed spaces.

$L : X \to Y$ linear operator.

linear : $L(\alpha x_1 + \beta x_2) = \alpha L(x_1) + \beta L(x_2)$

$\forall x_1, x_2 \in X, \forall \alpha, \beta \in \mathbb{R}.$

DEF

L is said to be bounded if

$\exists M \geq 0$ s.t. $\|Lx\|_Y \leq M \|x\|_X \quad \forall x \in X$

THEOREM. X, Y normed spaces, $L : X \to Y$ is linear.

Then the following statements are equivalent.

a) L is bounded.

b) L is continuous:

$\forall x_0 \in X, \forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$\|x - x_0\|_X < \delta \Rightarrow \|Lx - Lx_0\|_Y < \varepsilon$

c) L is continuous at one point.

Proof. a) $\Rightarrow$ b).

Fix $x_0 \in X$, Fix $\varepsilon > 0$.

$$\|Lx - Lx_0\|_Y = \|L(x - x_0)\|_Y \leq M \|x - x_0\|_X < M \frac{\varepsilon}{M} = \varepsilon$$

so, if $\|x - x_0\| < \dfrac{\varepsilon}{M} = \delta$.

b) $\Rightarrow$ c) trivial

c) $\Rightarrow$ a) Assume that L is continuous at $x_0 \in X$.

$\Rightarrow \forall \varepsilon>0\ \exists \delta>0$ s.t. $\|Lx - Lx_0\|_Y = \|L(x-x_0)\|_Y < \varepsilon$
if $\|x-x_0\|_X < \delta$

I set $x - x_0 = h$, so we know that

$\forall \varepsilon>0\ \exists \delta>0$ s.t. if $\|h\|_X < \delta \Rightarrow \|Lh\| < \varepsilon$.

Assume now that $x \in X \setminus \{0\}$ (for $x=0$ it is trivial)

$$\|Lx\|_Y = \left\|L\left(\frac{x}{\|x\|_X}\frac{\delta}{2}\frac{\|x\|_X 2}{\delta}\right)\right\|_Y =$$

$$= \frac{\|x\|}{\delta}2 \left\|L\left(\frac{x}{\|x\|}\frac{\delta}{2}\right)\right\|_Y < \frac{\varepsilon\|x\|2}{\delta} = \underbrace{\frac{2\varepsilon}{\delta}}_{= M}\|x\|$$

this has norm $\frac{\delta}{2} < \delta$

I have proved that $\|Lx\|_Y \leq M\|x\|_X \quad \forall x \in X$ ~~$\setminus\{0\}$~~

$\square$

Example of a linear operator which is not continuous.

$X = C([-1,1])$ with the $\|f\|_1 = \int_{-1}^{1} |f(x)|\,dx$

this a normed space

$Y = \mathbb{R}$

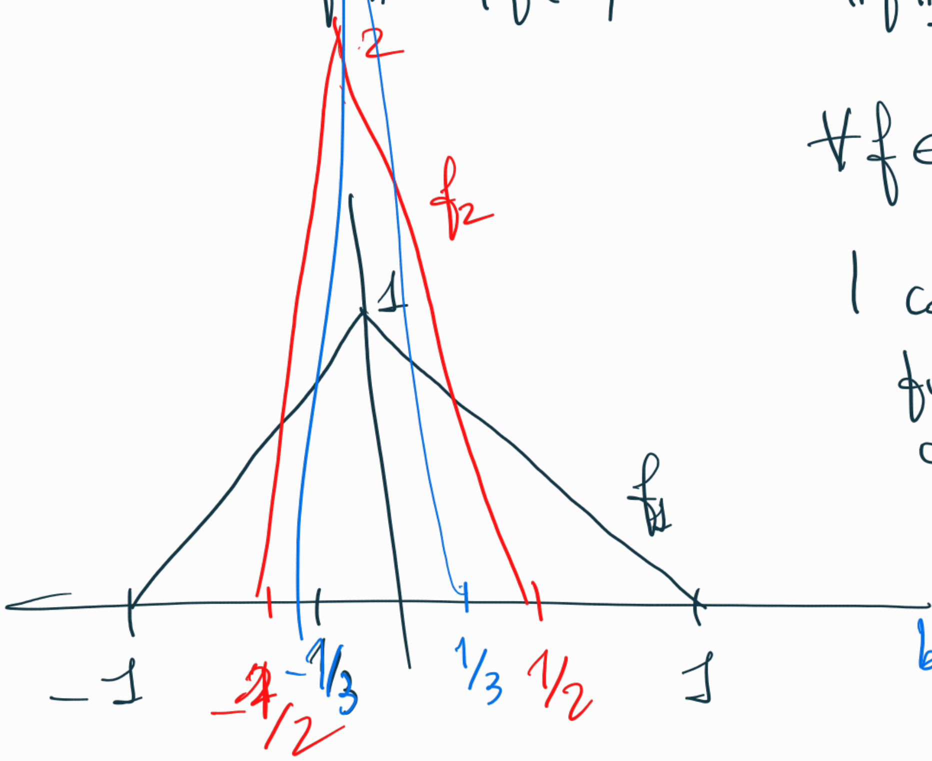
$L(f) = f(0)$ is clearly linear,

But it is not continuous.

If it were continuous, there should exist a constant M such that

$$\|L(f)\| = |f(0)| \leq M \, \|f\|_1 = M \int_{-1}^{1} |f(x)| \, dx$$

$$\forall f \in C([-1,1])$$



I can take a sequence f_n of continuous functions of L^1 norm $= 1$.

such that $f_n(0) \to +\infty$

but $f_n(0) = n \to +\infty$

The family $\mathcal{L}(X;Y) = \{L : X \to Y \text{ linear and continuous}\}$ is itself a vector space, which can be given a norm

$$\|L\| = \sup_{x \in X \setminus \{0\}} \frac{\|Lx\|_Y}{\|x\|_X} = \sup_{\substack{x \in X \\ \|x\|_X = 1}} \|Lx\|_Y$$

Theorem

X, Y are Banach spaces $\Rightarrow \mathcal{L}(X,Y)$ is a Banach space

If $Y=\mathbb{R}$, the operators in $\mathcal{L}(X,\mathbb{R})$ are called functionals

$$\mathcal{L}(X;\mathbb{R}) = X^*$$

$$X^* = \{ L : X \to \mathbb{R} \quad \text{linear and continuous} \}$$

Suppose $X = \mathbb{R}^N$

$$L \in (\mathbb{R}^N)^* \Leftrightarrow L : \mathbb{R}^N \to \mathbb{R} \quad \text{linear (and continuous)}$$

$$\Downarrow$$

$$\exists\, \xi \in \mathbb{R}^N \text{ s.t. } \quad Lx = \xi \cdot x$$

THEOREM (Riesz-Frechet representation theorem)

Let H be a Hilbert (scalar product + complete)

let $L : H \to \mathbb{R}$ linear continuous $(L \in H^*)$

$$\Rightarrow \exists!\, \xi \in H \text{ s.t. } \quad Lu = (\xi, u) \quad \forall u \in H$$

Proof: 1) existence of ξ let $M = L^{-1}(0) = \{ y \in H : Ly = 0 \}$

M closed subspace of H.

If $y_n \in M$, $y_n \to y$ $\overset{?}{\Longrightarrow}$ $y \in M$.

$y_n \in M \Rightarrow Ly_n = 0 \Rightarrow Ly = 0$ by continuity.

Two possibilities: 1) $M=H \Rightarrow L\equiv 0 \Rightarrow$ we take $\xi=0$

2) $M\subset H \Rightarrow \exists z\in M^\perp\setminus\{0\}$.

It suffices to take any $y\in H\setminus M$ and take orthogonal projections $y=\underbrace{Py}_{\in M}+\underbrace{Qy}_{\in M^\perp}$ different from zero

By replacing z by $\frac{z}{\|z\|}$, I can assume that $\|z\|=1$

Then we take $\xi=(Lz)z$

I want to prove that $\boxed{Lu=(\xi,u)=(Lz)(z,u)}$ $\forall u\in H$

I define $w=(Lu)z-(Lz)u$

I claim that $\underline{w\in M} \Longleftrightarrow Lw=0$

$$Lw=L\big((Lu)z-(Lz)u\big)=(Lu)(Lz)-(Lz)(Lu)=0$$

$$z\in M^\perp \longrightarrow (z,w)=0$$

$$\rightarrow (z,(Lu)z-(Lz)u)=0$$

$$(Lu)\underbrace{(z,z)}_{=\|z\|^2=1}-(Lz)(z,u)=0$$

$$\Rightarrow Lu=(Lz)(z,u)$$

2) Uniqueness of ξ.

Assume that exist $\xi_1, \xi_2 \in H$ s.t.

$$Lu = (\xi_1, u) = (\xi_2, u) \qquad \forall u \in H$$

$$\Longrightarrow (\xi_1 - \xi_2, u) = 0 \qquad \forall u \in H$$

Take $u = \xi_1 - \xi_2$

$$\Longrightarrow \underbrace{(\xi_1 - \xi_2, \xi_1 - \xi_2)}_{= \|\xi_1 - \xi_2\|^2} = 0 \qquad \Longrightarrow \xi_1 = \xi_2$$

$\square$

Weak solutions of the differential problem

$$(P) \quad \begin{cases} -u''(x) + u(x) = f(x) & \text{in } (a,b) \\ u(a) = u(b) = 0 \end{cases}$$

u is called weak sol^n. of (P) if.

1) $\underline{u \in W_0^{1,2}(a,b)} = H_0^1(a,b) =$

$$= \{ u \in W^{1,2}(a,b) = H^1(a,b) : \underbrace{\tilde{u}}_{\text{continuous representative of } u}(a) = \tilde{u}(b) = 0 \}$$

$$= \{ u \in W^{1,2}(a,b) : \exists \varphi_n(x) \in C_c^1(a,b) \text{ s.t. }$$

$$\varphi_n \to u \text{ in } W^{1,2}(a,b) \}$$

$$\begin{cases} \varphi_n \to u & \text{in } L^2(a,b) \\ \varphi_n' \to u' & \text{in } L^2(a,b) \end{cases}$$

and

2) $$\int_a^b u'\varphi'\,dx + \int_a^b u\varphi\,dx = \int_a^b f\varphi\,dx \quad \forall \varphi \in C_c^1(a,b)$$

Now, if $v \in H_0^1(a,b) = W_0^{1,2}(a,b)$

$\exists\, \varphi_n \in C_c^1(a,b)$ s.t. $\varphi_n \to v$ $L^2(a,b)$

$\varphi_n' \to v'$ "

$$\int_a^b u'\varphi_n'\,dx + \int_a^b u\varphi_n\,dx = \underbrace{\int_a^b f\varphi_n\,dx}$$

$n \to +\infty$ $\qquad n \to +\infty$

$$\int u'v'\,dx \qquad \int_a^b uv\,dx \qquad \int_a^b fv\,dx$$

$$\left|\int_a^b f\varphi_n\,dx - \int_a^b fv\,dx\right| = \left|\int_a^b f(\varphi_n - v)\,dx\right| =$$

$$= \left|(f, \varphi_n - v)_{L^2}\right| \leq \|f\|_{L^2}\, \underbrace{\|\varphi_n - v\|_{L^2}}_{\downarrow\, 0} \to 0$$

So, we have

$$\underbrace{\int_a^b u'v'\,dx + \int_a^b uv\,dx}_{(u,v)_{H_0^1}} = \int_a^b fv\,dx \quad \forall v \in W_0^{1,2}(a,b) = H_0^1(a,b)$$

So, we are looking for $u \in H_0^1$ s.t.

$$(u,v)_{H_0^1} = \underbrace{\int_a^b f v \, dx}_{= L(v)} \qquad \forall v \in H_0^1$$

$$L(v) : H_0^1(a,b) \longrightarrow \mathbb{R}$$

$$v \longmapsto L(v) = \int_a^b f v \, dx = (f,v)_{L^2}$$

$$\boxed{\begin{array}{l} L \text{ is linear.} \\ L \text{ is continuous in } H_0^1(a,b) \end{array}}$$

$$|Lv| = |(f,v)_{L^2}| \leq \|f\|_{L^2} \underset{\|v\|_{H_0^1}}{\underset{\wedge}{\|v\|_{L^2}}} \leq \|f\|_{L^2} \|v\|_{H_0^1}$$

So, we are looking for $u \in H_0^1(a,b)$ such that

$$(u,v)_{H_0^1} = Lv \qquad \forall v \in H_0^1 \quad (2)$$

By Riesz-Frechet's theorem $\exists !\ u \in H_0^1(a,b)$ such that (2) holds

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THEOREM $\forall f \in L^2(a,b)$ $\exists !\ u \in H_0^1(a,b)$ weak solution of (P).

We wish to repeat this process in dimension $N > 1$.

$u(x,y)$: Ω bounded open set in $\mathbb{R}^N$
$u(x,y,z)$

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\Delta u = \nabla^2 u = u_{xx} + u_{yy} \, (+ u_{zz}) =$$

$$= \sum_{i=1}^{N} u_{x_i x_i}$$

$$\nabla^2 u = \nabla \cdot (\nabla u) = \nabla \cdot (\text{grad}\, u) = \text{div}(\nabla u)$$

I take the equation $-\Delta u + u = f$ and I multiply it by a test function $\varphi \in C_c^1(\Omega)$

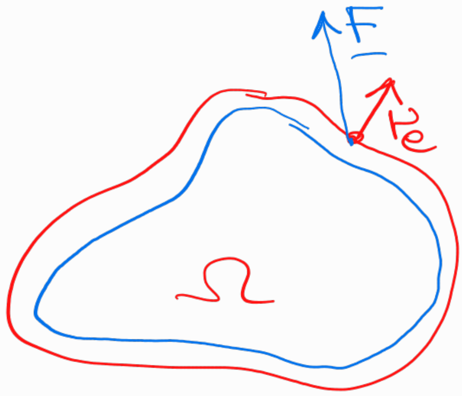
$$\underbrace{\int_\Omega (-\Delta u)\varphi \, dx}_{\substack{\shortparallel \\ -\int (\text{div}\, \nabla u)\varphi \, dx}} + \int_\Omega u\varphi \, dx = \int_\Omega f\varphi \, dx$$

Divergence theorem Ω regular open set, $\underline{F} \in C^1(\overline{\Omega})$ vector field.

$$\Rightarrow \int_\Omega \text{div}\, \underline{F} \, dx = \int_{\partial\Omega} \underline{F} \cdot \underline{\nu}_e \, d\sigma$$

flux of F exiting from $\partial\Omega$.

$$\operatorname{div} \underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \left(+ \frac{\partial F_3}{\partial z} \right)$$



$$\int_\Omega \Delta u \; \varphi \, dx = \int_\Omega \operatorname{div}(\nabla u) \varphi \, dx =$$

$$= \int_\Omega \operatorname{div}(\nabla u \varphi) \, dx - \int_\Omega \nabla u \cdot \nabla \varphi \, dx \overset{(\text{div. thm})}{=}$$

$$\left[\begin{aligned} \operatorname{div}(\nabla u \varphi) &= \sum_i (u_{x_i} \varphi)_{x_i} = \\ &= \sum_i u_{x_i x_i} \varphi + \sum_i u_{x_i} \varphi_{x_i} = \\ &= \operatorname{div}(\nabla u) \varphi + \nabla u \cdot \nabla \varphi \end{aligned} \right]$$

$$= \underbrace{\int_{\partial\Omega} \nabla u \varphi \cdot \nu_e \, d\sigma}_{} - \int_\Omega \nabla u \cdot \nabla \varphi \, dx = - \int_\Omega \nabla u \cdot \nabla \varphi \, dx$$

$$\left(\nabla u \cdot \nu_e = \frac{\partial u}{\partial \nu_e} \right)$$

$$\int_{\partial\Omega} \varphi \frac{\partial u}{\partial \nu_e} \, d\sigma = 0$$

because φ is zero near $\partial\Omega$.

Therefore, the equality

$$\underbrace{\int_\Omega (-\Delta u)\varphi\, dx}_{\parallel \atop -\int (\mathrm{div}\, \nabla u)\varphi\, dx} + \int_\Omega u\varphi\, dx = \int_\Omega f\varphi\, dx$$

becomes

$$\int_\Omega \nabla u \cdot \nabla\varphi\, dx + \int_\Omega u\varphi\, dx = \int_\Omega f\varphi\, dx \quad \forall \varphi \in C_c^1(\Omega)$$

Where is u?

We must define <u>Sobolev spaces in dim $N > 1$.</u>

Let $u \in L^p(\Omega)$ $\qquad 1 \le p \le \infty$

I will say that $v_i \in L^p(\Omega)$ $\quad (i = 1, 2, \dots N)$

is the i-th weak derivative of u if

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega v_i \varphi\, dx \quad \forall \varphi \in C_c^1(\Omega)$$

This is motivated by the following:

If $u \in C^1(\overline{\Omega})$ we have

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega \frac{\partial u}{\partial x_i}\varphi\, dx \qquad \forall \varphi \in C_c^1(\Omega) \quad (3)$$

Proof

$$\underline{F} = (f, 0, \dots 0) \qquad \operatorname{div} F = \frac{\partial f}{\partial x_1}$$

Divergence theorem $\Rightarrow$

$$\int_\Omega \frac{\partial f}{\partial x_1}\, dx = \int_{\partial\Omega} \underline{F} \cdot \nu_e \, d\sigma = \int_{\partial\Omega} f (\nu_e)_1 \, d\sigma$$

If I apply it to a product. $f \to u\varphi$

$$\int_\Omega \frac{\partial u}{\partial x_1} \varphi \, dx + \int_\Omega u \frac{\partial \varphi}{\partial x_1} dx = \int_{\partial\Omega} u\varphi (\nu_e)_1 \, d\sigma = 0$$

if $\varphi = 0$ on $\partial\Omega$

$$\Rightarrow \int_\Omega \frac{\partial u}{\partial x_1} \varphi \, dx = - \int_\Omega u \frac{\partial \varphi}{\partial x_1} dx \qquad \forall \varphi \in C^1_c(\Omega)$$

<u>DEF.</u> Sobolev Space $W^{1,p}(\Omega)$ in dim $N \geq 1$.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded, open set in $\mathbb{R}^N$.

$1 \leq p \leq \infty$.

$$W^{1,p}(\Omega) = \Big\{ u \in L^p(\Omega) \text{ which admits weak partial derivatives } v_1, v_2, \dots v_N \in L^p(\Omega) \text{ s.t. } \int_\Omega u \, \varphi_{x_i} \, dx = - \int_\Omega v_i \varphi \, dx \quad \forall \varphi \in C^1_c(\Omega) \Big\}$$

as before, we will denote the weak derivatives

v_i by $\dfrac{\partial u}{\partial x_i} = u_{x_i}$

and $(v_1, v_2, \dots v_N)$ by ∇u.

This is a vector space, and it is a normed space with the norm.

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\,|\nabla u|\,\|_{L^p(\Omega)}$$

$$= \|u\|_{L^p(\Omega)} + \left[\int_\Omega |\nabla u|^p \, dx\right]^{1/p}$$

or we could take

$$\|u\|_{W^{1,p}(\Omega)} = \left[\int_\Omega \left(|u|^p + \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^p\right) dx\right]^{1/p}$$

These norms are equivalent. The space $W^{1,p}(\Omega)$ is a Banach space

In the special case $p=2$ $W^{1,p}(\Omega)$ is a Hilbert space with the scalar product.

$$(u,v)_{W^{1,2}(\Omega)} = \int_\Omega u v \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx =$$

$$= (u,v)_{L^2(\Omega)} + \sum_{i=1}^{N} (u_{x_i}, v_{x_i})_{L^2(\Omega)}$$