We wish to study a problem of the form

$$\begin{cases} -\Delta u(x) + u(x) = f(x) & x \in \Omega \text{ open set in } \mathbb{R}^N \\ \quad \shortparallel \\ -\nabla^2 u \\ \quad \shortparallel \\ -(u_{xx} + u_{yy} + u_{zz}) & u \in C^2(\Omega) \cap C(\overline{\Omega}) \\ u(x) = 0 \quad \text{on } \partial\Omega. & \text{(Dirichlet problem).} \end{cases}$$

The plan is the following:

1) We give a notion of a weak solution (in an integral form) such that every classical solution is also a weak solution.

2) Functional analysis $\Rightarrow$ under weak hypotheses on f there exists a unique weak solution. u

3) If f is "more regular", then u is "more regular" too.
for instance $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$

4) If u is a weak solution of class $C^2(\Omega) \cap C^0(\overline{\Omega})$, then it is also a classical solution.

Let us try this in dim. 1. The problem becomes

$$\begin{cases} -u''(x) + u(x) = f(x) & \text{in } (a,b) \\ u(a) = u(b) = 0 \end{cases}$$

The first aim is to solve Step 1 $\Rightarrow$ to give a notion of weak solution.
To this aim we defined the Sobolev spaces

<u>DEF</u> we say that $u \in W^{1,p}(a,b)$ $\quad 1 \le p \le \infty$
if $u \in L^p(a,b)$ and $\exists\, v \in L^p(a,b)$ (weak derivative) such that

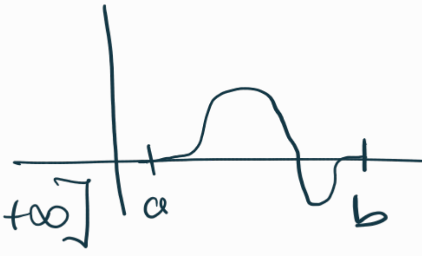
$$\int_a^b u(x)\varphi'(x)\,dx = -\int_a^b v(x)\varphi(x)\,dx \quad \forall \varphi(x) \in C^1_c(a,b)$$

$$\shortparallel$$

$$-\int_a^b u'(x)\,\varphi(x)\,dx$$

we will often write $v = u'$

functions C^1 such that they are zero near a and b



Of course, if $u \in C^1([a,b])$, then it is also in $W^{1,p}(a,b)$ $\quad \forall p \in [1,+\infty]$

Rem.

$W^{1,p}(a,b)$ is a vector space.

If $u, w \in W^{1,p}(a,b) \Rightarrow \alpha u + \beta w \in W^{1,p}(a,b)$ $\forall \alpha, \beta \in \mathbb{R}$

and $(\alpha u + \beta w)' = \alpha u' + \beta w'$

On $W^{1,p}(a,b)$ we introduce a norm.

$$\|u\|_{W^{1,p}(a,b)} = \|u\|_{L^p} + \|u'\|_{L^p}$$ this is a norm on $W^{1,p}(\Omega)$

we can also choose a different norm

$$|||u|||_{W^{1,p}(a,b)} = \left(\|u\|_{L^p}^p + \|u'\|_{L^p}^p\right)^{1/p} =$$

$$= \left[\int_a^b |u|^p\,dx + \int_a^b |u'|^p\,dx\right]^{1/p} =$$

$$= \left[\int_a^b \left(|u|^p + |u'|^p\right) dx\right]^{1/p}$$

These are equivalent norms, in the sense that there exist positive constants c_1, c_2 s.t.

$$c_2 \|u\|_{W^{1,p}} \leq |||u|||_{W^{1,p}} \leq c_1 \|u\|_{W^{1,p}}$$

So converging sequences are the same in the two norms, open sets of $W^{1,p}$ are the same.

It is easy to see that these are norms, in particular the triangle inequality holds

$$\|u+v\|_{W^{1,p}} \leq \|u\|_{W^{1,p}} + \|v\|_{W^{1,p}}$$

<u>Remark</u> The weak derivative is unique, up to changes in sets with measure zero.

Suppose that $u \in W^{1,1}(a,b)$ has two weak derivatives $v, w \in L^1(a,b)$ s.t.

$$\left.\begin{aligned} &\int_a^b u\varphi'\,dx = -\int_a^b v\,\varphi\,dx \\ \text{and } &\int \quad '' \quad = -\int_a^b w\,\varphi\,dx \end{aligned}\right| \Longrightarrow$$

$$\Longrightarrow \int_a^b v\varphi\,dx = \int_a^b w\varphi\,dx$$

$$\Longrightarrow \int_a^b \underbrace{(v-w)}_{g}\varphi\,dx = 0 \qquad \forall \varphi \in C_c^1(a,b)$$

Fundamental lemma of Calculus of Variations.

If $g \in L^1(a,b)$ is such that $\int_a^b g\varphi\,dx = 0 \quad \forall \varphi \in C_c^1(a,b)$

then $g \equiv 0$ a.e. on (a,b).

$\Longrightarrow v = w$ a.e. on (a,b) $\square$

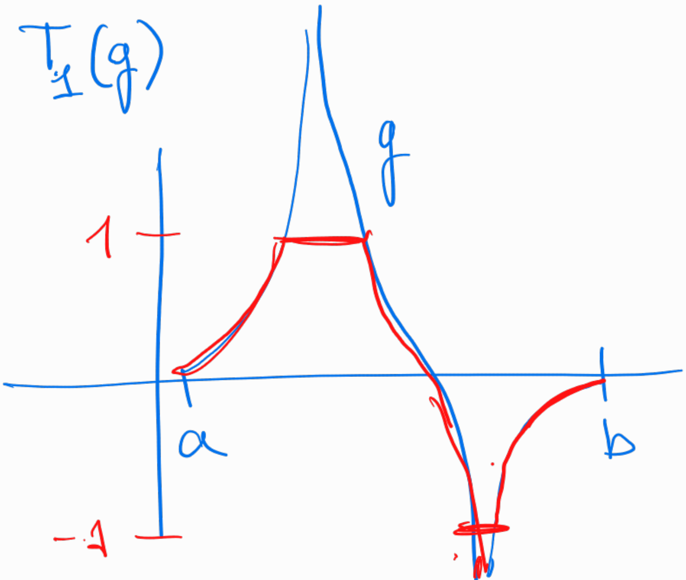
Proof of fundamental lemma (incomplete)

The idea is to take $\varphi = T_1(g)$

where

$T_1(g) = \max\{\min\{g(x), 1\}, -1\}$

$T_1(g) \in L^\infty(\Omega)$



Of course $\boxed{T_1(g) \notin C^1_c(a,b)}$ But if I could take $\varphi = T_1(g)$ i would obtain

$$\int_a^b g\, T_1(g) = 0$$

$$\shortparallel$$

$$\int_a^b |g|\, |T_1(g)|$$

$$\rotatebox{90}{$\geq$}$$

$$\int_a^b |T_1 g|^2 \implies T_1 g \equiv 0 \quad a.e.$$

$$\implies g \equiv 0 \quad a.e.$$

So the second part of the proof is that

$$\boxed{\begin{array}{l} \text{we can find a sequence } \varphi_n \in C^1_c(a,b) \text{ such that} \\ \varphi_n(x) \to T_1(g) \quad a.e. \\ \|\varphi_n(x)\|_{L^\infty} \leq 1 \end{array}}$$

If we know this, we will show that

$$\int_a^b g\, \varphi_n(x)\, dx = 0$$

$$\downarrow$$

$$\int_a^b g\, T_1(g)\, dx$$

1) $g\, \varphi_n(x) \to g\, T_1(g)$ a.e.

2) $|g\, \varphi_n(x)| \leq |g| \in L^1(a,b)$

\+ dominated convergence thm.

References for Sobolev Spaces:

H. Brezis: Functional Analysis

L.C. Evans: Partial Differential Equation

THEOREM $W^{1,p}(a,b)$ is a Banach space, i.e. it is complete

Proof: let $\{u_n\}$ be a Cauchy sequence in $W^{1,p}(a,b)$. I want to prove that it converges in $W^{1,p}(a,b)$ to some $u \in W^{1,p}(a,b)$.

$\{u_n\}$ is Cauchy $\Rightarrow$

$$\forall \varepsilon>0 \ \exists \bar{n} \text{ s.t. } \forall m,n>\bar{n} \quad \|u_n-u_m\|_{W^{1,p}} < \varepsilon$$

$$\|u_n-u_m\|_{W^{1,p}} = \|u_n-u_m\|_{L^p} + \|u_n'-u_m'\|_{L^p}$$

$\rightarrow$ $\{u_n\}$ is a Cauchy sequence in $L^p(a,b)$

$\{u_n'\}$ is a Cauchy sequence in $L^p(a,b)$

L^p is complete $\Longrightarrow$ $u_n \to u$ in L^p

$u_n' \to v$ in L^p

Remark $u_n \to u$ in $W^{1,p}(a,b)$ means

$u_n \to u$ in L^p, $u_n' \to u'$ in L^p

We only have to show that $v = u'$

we know that

$$\int_a^b u_n \varphi' dx = - \int_a^b u_n' \varphi \, dx \qquad \forall \varphi \in C_c^1(a,b)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\int_a^b u\varphi' dx \qquad - \int_a^b v\varphi \, dx \quad \Rightarrow u' = v$$

Let us prove this convergence, the other is similar.

$$\left| \int_a^b (u_n \varphi' - u\varphi') dx \right| = \left| \int_a^b (u_n - u)\varphi' dx \right| \leq$$

Hölder's ineq.

$$\leq \int_a^b \underbrace{|u_n - u|}_{\in L^p} \underbrace{|\varphi'|}_{\in L^\infty} dx \leq \underbrace{\|u_n - u\|_{L^p}}_{\to 0} \|\varphi'\|_{L^q} \longrightarrow 0$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

By definition, functions in $W^{1,p}(a,b)$ are L^p functions.
Actually, they are continuous in $[a,b]$

Theorem (Continuity of $W^{1,p}(a,b)$ functions)

$1 \leq p \leq \infty$. Let $u \in W^{1,p}(a,b)$.

$\Rightarrow \exists \, \tilde{u} \in C([a,b])$ s.t. $u = \tilde{u}$ a.e. on (a,b)

Moreover the following formula holds

$$\tilde{u}(x) - \tilde{u}(y) = \int_y^x u'(t) \, dt \qquad \text{(Fundamental theorem of Integral Calculus)}$$

Proof. We will prove it under the weakest assumption $p=1$.
Let $u \in W^{1,1}(a,b)$. We define

$$\bar{u}(x) = \int_a^x u'(t)\,dt \quad \text{(well defined because } u' \in L^1\text{)}$$

1) $\bar{u}$ is a continuous function on $[a,b]$.

$$x_n \to x \text{ in } [a,b] \overset{?}{\Rightarrow} \bar{u}(x_n) \to \bar{u}(x)$$

$$|\bar{u}(x_n) - \bar{u}(x)| = \left| \int_a^{x_n} u'(t)\,dt - \int_a^{x} u'(t)\,dt \right| = \left| \int_x^{x_n} u'(t)\,dt \right|$$

$$\le \left| \int_x^{x_n} |u'(t)|\,dt \right| =$$

Assume $x_n \to x^+$

$$= \int_x^{x_n} |u'(t)|\,dt = \int_a^b \underbrace{|u'(t)|\,\chi_{(x,x_n)}}_{\to 0 \text{ a.e.}} \xrightarrow{n\to\infty} 0.$$

$\le |u'(t)| \in L^1(a,b)$

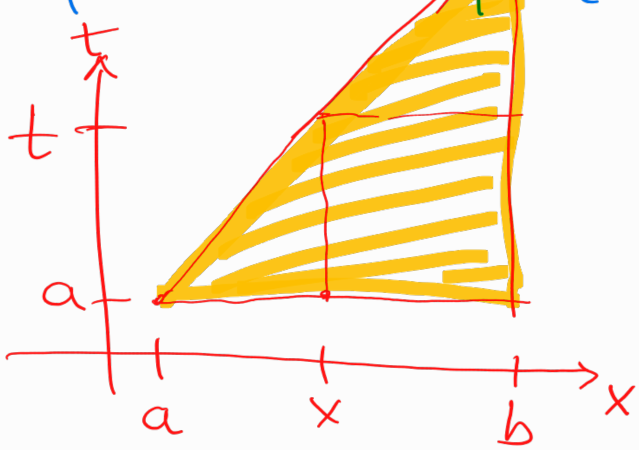
2) We want to show that u' is the weak derivative of $\bar{u}$, i.e.

$$\int_a^b \bar{u}(x)\varphi'(x)\,dx \overset{?}{=} -\int_a^b u'(x)\varphi(x)\,dx \qquad \forall \varphi \in C_c^1(a,b)$$

$$\parallel$$

$$\int_a^b \left(\int_a^x u'(t)\,dt \right) \varphi'(x)\,dx$$

$\parallel$ Fubini theorem



$$\int_a^b dt \left[\int_t^b dx \; u'(t)\,\varphi'(x)\right]$$

because
$u'(t)\,\varphi'(x) \in L^1((a,b)^2)$
$\cap \qquad \cap$
$L^1(a,b) \; L^\infty(a,b)$

$$= \int_a^b dt \; u'(t) \Bigg[\underbrace{\int_t^b \varphi'(x)\,dx}_{\varphi(b)-\varphi(t)}\Bigg] = -\int_a^b u'(\cancel{t}_x)\,\varphi(\cancel{t}_x)\,d\cancel{t}_x$$

$\varphi(b)-\varphi(t)$
$\parallel$
0

just as desired.

3) conclusion.

u and $\bar{u}$ have the same weak derivative.

$$\Rightarrow \left.\begin{aligned} \int_a^b u\varphi'\,dx &= -\int_a^b u'\varphi\,dx \\ \int_a^b \bar{u}\varphi'\,dx &= -\int_a^b u'\varphi\,dx \end{aligned}\right\} \quad \forall \varphi \in C_c^1(\Omega)$$

$$\Rightarrow \int_a^b \underbrace{(u-\bar{u})}_{f}\varphi'\,dx = 0 \qquad \forall \varphi \in C_c^1(\Omega)$$

Lemma If $f \in L^1(a,b)$ satisfies

$$\boxed{\int_a^b f\varphi'\,dx = 0 \quad \forall \varphi \in C_c^1(\Omega)}$$

the weak derivative of f is zero

$\Rightarrow \exists\, c \in \mathbb{R}$ s.t. $f(x) = c$ a.e. on (a,b)

$$\Rightarrow u = \underbrace{\bar{u} + c}_{\tilde{u}} \qquad \text{a.e. on } (a,b)$$

$\tilde{u}$ is the desired function

$$\tilde{u}(x) - \tilde{u}(y) = \int_a^x u'(t)\,dt + \cancel{c} - \left(\int_a^y u'(t)\,dt + \cancel{c} \right) =$$

$$= \int_y^x u'(t)\,dt \qquad \square$$

<u>Remark.</u> Assume that $u_n \to u$ in $W^{1,1}(\Omega)$

$$|\tilde{u}_n(x) - \tilde{u}(x)| = \left| \int_a^x (u_n'(t) - u'(t))\,dt \right| \overset{\text{(triangle)}}{\leq}$$

$$\leq \int_a^x |u_n'(t) - u'(t)|\,dt \leq \int_a^b |u_n'(t) - u'(t)|\,dt =$$

$$= \|u_n' - u'\|_{L^1} \leq \|u_n - u\|_{W^{1,1}(a,b)}$$

$$\Rightarrow \quad \max_{[a,b]} |\tilde{u}_n(x) - \tilde{u}(x)| \leq \|u_n - u\|_{W^{1,1}} \longrightarrow 0$$

this means

$u_n \to u$ in $W^{1,1}(a,b) \Rightarrow$ $u_n \to u$ in $L^\infty(a,b)$

$u_n \to u$ uniformly on $[a,b]$

<u>DEF</u> Let $1 \leq p \leq \infty$.

We define $W_0^{1,p}(a,b)$ as "the closure in the $W^{1,p}$-norm of the set $C_c^1(a,b)$"

$$W_0^{1,p}(a,b) = \{ u \in W^{1,p}(a,b) \text{ s.t. } \exists \varphi_n \in C_c^1(a,b)$$

s.t. $\varphi_n \to u$ in $W^{1,p}(a,b)\}$

THEOREM

$$W_0^{1,p}(a,b) = \{u \in W^{1,p}(a,b) \text{ s.t. } \tilde{u}(a) = \tilde{u}(b) = 0\}$$

continuous representative of u

$W_0^{1,p}(a,b)$ is a subspace of $W^{1,p}(a,b)$
it is a closed subspace

if $u_n \to u$ in $W^{1,p}(a,b) \Rightarrow \tilde{u}_n(x) \to \tilde{u}(x)\ \forall x \in [a,b]$
$\rightarrow \tilde{u}(a) = 0,\ \tilde{u}(b) = 0$
$\Rightarrow u \in W_0^{1,p}(a,b)$

The "best" Sobolev spaces are where $p=2$

$$W^{1,2}(a,b) = \{u \in L^2(a,b) : u' \in L^2(a,b)\}$$

$$W_0^{1,2}(a,b) = \{u \in W^{1,2}(a,b) : \tilde{u}(a) = \tilde{u}(b) = 0\}$$

These are Hilbert spaces with the scalar product

$$(u,v) = \int_a^b u(x)v(x)\,dx + \int_a^b u'(x)v'(x)\,dx$$

Now let us give the def. of weak solution of problem

$$(P) \quad \begin{cases} -u'' + u = f & \text{in } (a,b) \\ u(a) = u(b) = 0 \end{cases}$$

DEF. A weak solution of (P) is a function $u \in W_0^{1,2}(a,b)$ s.t.

$$\int_a^b u'(x)\varphi'(x)\,dx + \int_a^b u(x)\varphi(x)\,dx = \int_a^b f(x)\varphi(x)\,dx \qquad \forall \varphi \in C_c^1(a,b).$$

Let us check that a classical solution of (P) is also a weak solution.

Let $\underbrace{u(x) \in C^2([a,b])}$ a classical solution of P.

$$\Downarrow$$

$$u \in W_0^{1,2}(a,b)$$

We multiply the equation by $\varphi(x) \in C_c^1(a,b)$ and integrate.

$$\underbrace{-\int_a^b u''(x)\varphi(x)\,dx}_{\|} + \int_a^b u(x)\varphi(x)\,dx = \int_a^b f(x)\varphi(x)\,dx$$

$$\int_a^b u'(x)\varphi'(x)\,dx - \cancel{u'(b)\varphi(b)} + \cancel{u'(a)\varphi(a)}$$

So, a classical solution is also a weak solution.

$$\boxed{\begin{array}{l}\textbf{Step 2} \quad \exists!\ \text{weak solution of (P)}\ \forall f \in L^2(a,b) \\ \text{To be proved next time with Functional Analysis}\end{array}}$$

Step 3 Regularity.

If $f \in C([a,b])$, then the weak solution u

$\cap$
$L^2(a,b)$

(which exists by Step 2) is $C^2([a,b])$.

Proof. We know that

$$\int_a^b u'\varphi'\,dx = -\int_a^b u\varphi + \int_a^b f\varphi =$$

$$= -\int_a^b \underbrace{(u-f)}_{\text{continuous}}\varphi \qquad \forall \varphi \in C_c^1(a,b)$$

$\Rightarrow$ The weak derivative of u' is $u-f$ (continuous)

$$u'(x) = C + \int_a^x \underbrace{\underbrace{(u')'(t)}_{(u-f)(t)}}_{\text{continuous}}\,dt \Rightarrow u'(x) \in C^1([a,b])$$

$$\Downarrow$$

$$u \in C^2([a,b])$$

Step 4 If u is a weak solution of (P) of class $C^2([a,b])$, then it is a classical solution

$$\underbrace{\int_a^b u'\varphi'\,dx}_{= -\int_a^b u''\varphi\,dx} + \int_a^b u\varphi\,dx = \int_a^b f\varphi\,dx$$

$$\Longrightarrow \quad -\int_a^b u'' \varphi \, dx + \int_a^b u \varphi \, dx = \int_a^b f \varphi \, dx$$

$$\Longrightarrow \quad \int_a^b (-u'' + u - f) \varphi \, dx = 0 \qquad \forall \varphi \in C_c^1(a,b)$$

$$\Longrightarrow \quad \underbrace{-u'' + u - f}_{\text{continuous}} = 0 \qquad \text{a.e.}$$

$$\Longrightarrow \quad -u'' + u - f = 0 \qquad \forall x \in (a,b). \qquad \square$$