Let H be a Hilbert space (Scalar product + complete)
Let $\{u_1, \dots u_k\}$ be an orthonormal set.
Then, for any fixed $x \in H$,
the linear combination

$$s_k = \sum_{j=1}^{k} (x, u_j)\, u_j$$

j-th Fourier coefficient

minimizes the distance of x from all linear combinations of $\{u_1, \dots u_k\}$:

$$\| x - s_k \| = \min \left\{ \|x - s\| : s = \sum_{j=1}^{k} c_j u_j \right\}$$

Moreover one has

$$\| s_k \|^2 = \sum_{j=1}^{k} (x, u_j)^2 \leq \|x\|^2$$

Bessel's inequality.

In the case where $H = L^2(-\pi, \pi)$, endowed with the scalar product

$$(f, g) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, g(t)\, dt$$

we take the orthonormal set

$$u_0(t) \equiv \frac{\sqrt{2}}{2}, \qquad u_j(t) = \cos(jt)$$

$$v_j(t) = \sin(jt) \qquad\qquad j = 1, \dots n$$

(that is, $k = 2n + 1$)

and we obtain that the best coefficients for $f \in L^2(-\pi,\pi)$ are

$$\tilde{a}_0 = (f, u_0) = \frac{\sqrt{2}}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$a_j = (f, u_j) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(jt)\,dt$$

$$b_j = (f, v_j) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(jt)\,dt$$

And the best approximation of f in $L^2(-\pi,\pi)$ for any choice of $n \in \mathbb{N}$ is

$$S_n(t) = \underbrace{\tilde{a}_0 \frac{\sqrt{2}}{2}} + \sum_{j=1}^{n} a_j \cos(jt) + b_j \sin(jt) =$$

$$\parallel$$

$$\frac{\sqrt{2}}{2}\cdot \underbrace{\frac{\sqrt{2}}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt}_{\tilde{a}_0}$$

$$\parallel$$

$$\frac{1}{2}\Big[\underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt}_{a_0}\Big]$$

$$= \frac{a_0}{2} + \sum_{j=1}^{n}\Big[a_j\cos(jt) + b_j\sin(jt)\Big]$$

And <u>Bessel's inequality</u> takes the form

$$\frac{a_0^2}{2} + \sum_{j=1}^{n}\left(a_j^2 + b_j^2\right) \leq \|f\|_{L^2}^2$$

and, letting $n \to +\infty$

$$\frac{a_0^2}{2} + \sum_{j=1}^{\infty}(a_j^2+b_j^2) \leq \|f\|_{L^2}^2$$

Complex notation

$H = L^2((-\pi,\pi);\mathbb{C})$, with the scalar product

$$(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\bar{g}(t)\,dt$$

$$\|f\|_2^2 = (f,f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\,dt$$

We take the orthonormal system

$$\boxed{u_n(t) = e^{int}} = \cos(nt) + i\sin(nt) \qquad n \in \underset{\mathbb{Z}}{\cancel{\mathbb{N}}}$$

$$\|u_n\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{|e^{int}|^2}_{\substack{\shortparallel \\ 1}}\,dt = 1$$

$$(u_n,u_m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\,\overline{(e^{imt})}\,dt = \qquad n \neq m.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\,e^{-imt}\,dt =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)t}\,dt = \frac{1}{2\pi i(n-m)}\,e^{i(n-m)t}\Big|_{-\pi}^{\pi} = 0$$

In this case, the theorem states.

For any fixed $f \in L^2(-\pi, \pi)$

$\exists!$ trig. polynomial

$$S_n(t) = \sum_{k=-n}^{n} c_k e^{+ikt}$$

where $c_k = (f, e^{ikt}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}$

which minimizes the distance from f.

Problem: What happens when $\underset{n}{k} \longrightarrow +\infty$.

For instance, in the case $H = L^2(-\pi, \pi)$

$$\|S_n - f\|_2 \xrightarrow[n \to +\infty]{} 0 \quad ?$$

that is, $\quad s_n \longrightarrow f$ in $L^2(-\pi, \pi)$?

Theorem Let $\{u_n\}_{n \in \mathbb{N}}$ be an orthonormal set in H. Then the following statements are equivalent.

1) $\{u_n\}$ is a maximal orthonormal set.

(i.e., you cannot find a non zero vector which is orthogonal to all u_n ?)

2) The set S of all finite linear combination of elements of $\{u_n\}$ is dense in H.

(i.e. $\forall x \in H$, $\forall \varepsilon > 0$ $\exists n \in \mathbb{N}$.

$$\exists \sum_{k=1}^{n} c_k u_k \quad s.t.$$

$$\left\| \sum_{k=1}^{n} c_k u_k - x \right\| < \varepsilon .)$$

3) $\forall x \in H \qquad x = \sum_{k=1}^{\infty} c_k u_k \quad$ with $\; c_k = (x, u_k)$

i.e. $\qquad \left\| \sum_{k=1}^{n} c_k u_k - x \right\| \xrightarrow{n \to +\infty} 0$

4) $\forall x \in H \qquad \|x\|^2 = \sum_{n=1}^{\infty} (x, u_n)^2 \qquad$ Parseval's identity

5) $\forall x, y \in H \qquad (x, y) = \sum_{n=1}^{\infty} (x, u_n)(y, u_n)$

The proof of this result could be an end-of-course seminar

(→ Rudin)

Theorem In $L^2(-\pi, \pi)$ the orthonormal system $\left\{ \frac{\sqrt{2}}{2}, \cos(nt), \sin(nt) \right\}_{n \in \mathbb{N}}$ is a maximal orthonormal system.

In the next lessons I would like to present and solve some problems related to partial differential equations. For instance, if we define the differential operator

$$\Delta u = \nabla^2 u = \nabla \cdot \nabla u = \operatorname{div}(\nabla u) =$$

$$= u_{x_1 x_1} + u_{x_2 x_2} + \dots + u_{x_n x_n} = \sum_{i=1}^{n} u_{x_i x_i}$$

we may consider the following problem.

$$(P)\begin{cases} -\Delta u(x) + u(x) = f(x) \ \leftarrow \text{given} & x \in \Omega \text{ bounded open set in } \mathbb{R}^N \\ u(x) = 0 & \text{on } \partial\Omega \leftarrow \text{Dirichlet Boundary condition.} \end{cases}$$

Other boundary conditions can be

$$u(x) = g(x) \leftarrow \text{given}$$

$$\frac{\partial u}{\partial \nu}(x) = h(x) \swarrow$$

Dimension $n=1$

$$\Omega = (a,b)$$

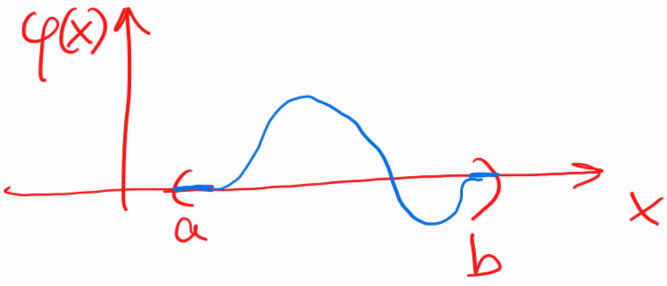
$$(P)\begin{cases} -u''(x) + u(x) = f(x) & \text{in } (a,b) \\ u(a) = u(b) = 0 \end{cases}$$

This problem is easily solved, but we use it to show our approach.

Le $u \in C^2([a,b])$ continuous up to the second derivative be a solution of problem (P).

Let $\varphi(x)$ be a function in $C^1_c(a,b)$

(that is $\varphi(x)$ is a $C^1(a,b)$ with compact support in (a,b), i.e. it must be zero near a and b)



I multiply the equation by $\varphi(x)$ and integrate on (a,b).

$$\underbrace{-\int_a^b u''(x)\varphi(x)\,dx}_{\substack{\| \text{ integrating by part.} \\ \int_a^b u'(x)\varphi'(x)\,dx \,-\, \underset{\overset{\|}{0}}{\cancel{\left[u'(x)\varphi(x)\right]_a^b}}}} + \int_a^b u(x)\varphi(x)\,dx = \int_a^b f(x)\varphi(x)\,dx$$

$\Longrightarrow$

$$(*) \int_a^b u'(x)\varphi'(x)\,dx + \int_a^b u(x)\varphi(x)\,dx = \int_a^b f(x)\varphi(x)\,dx$$

$$\forall \underset{\substack{\uparrow \\ \text{function test.}}}{\varphi} \in C_c^1(a,b)$$

Idea: take this last formula as a definition of a weak solution of problem (P).

We will say that u is a weak solution of (P) if $u \in X$ functional space to be identified.

and it satisfies the integral equality $(*)$ $\forall \varphi \in C_c^1(a,b)$

The program is the following.

1) we give a def^n of weak solution (identify X)

2) prove that there exists a unique weak solution

3) regularity. The weak solution is indeed regular (for instance, $u \in C^2([a,b])$.)

4) we prove that a weak solution of class $C^2([a,b])$ is indeed a classical solution

Let us prove that a C^2 function which satisfies (*) is a classical solution of the equation

continuous

$$(*) \quad \underbrace{\int_a^b u'(x)\varphi'(x)\,dx}_{=\text{ integrating by parts}} + \int_a^b u(x)\varphi(x)\,dx = \int_a^b f(x)\varphi(x)\,dx \qquad \forall \varphi \in C^1_c(a,b)$$

$$-\int_a^b u''(x)\varphi(x)\,dx + \cancel{u'(b)\varphi(b)} - \cancel{u'(a)\varphi(a)}$$

$$\Rightarrow \int_a^b \underbrace{\left(-u''(x)+u(x)-f(x)\right)}_{g(x)\in C([a,b])} \varphi(x)\,dx = 0 \qquad \forall \varphi \in C^1_c(a,b)$$

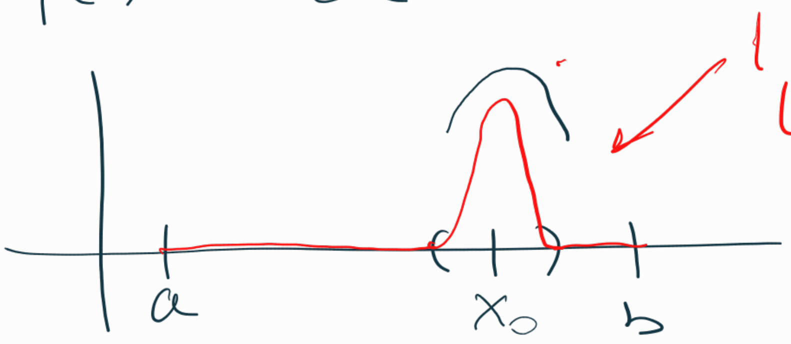
I claim that $g(x) \equiv 0$ in $[a,b]$.

Let us assume that $\exists\, x_0 \in [a,b]$ s.t. $g(x_0) \neq 0$

(for instance $g(x_0) > 0$)

$$\Rightarrow \quad g(x) \geq \frac{g(x_0)}{2} > 0 \qquad \forall x \in (x_0-\varepsilon, x_0+\varepsilon).$$

I take $\varphi(x) \in C^1_c(a,b)$



I choose φ like this!

$\varphi \geq 0$, $\quad \varphi > 0$ on $(x_0-\varepsilon, x_0+\varepsilon)$

$\varphi=0$ outside of this interval

$$0=\int_a^b g(x)\,\varphi(x)\,dx=\int_{x_0-\varepsilon}^{x_0+\varepsilon} g(x)\varphi(x)\,dx>0$$ absurd.

Sobolev Spaces. $I=(a,b)$ $\quad -\infty<a<b<+\infty$

Assume $u(x)\in C^1(I)$, let $\varphi(x)\in C^1_c(I)$. Then

$$\int_a^b u(x)\,\varphi'(x)\,dx \underset{\uparrow}{=} -\int_a^b u'(x)\,\varphi(x)\,dx \qquad (**)$$

integration by parts

We transform this equality into a definition

DEF Let $u(x)\in L^p(I)$ $\qquad 1\le p\le\infty$

(actually, $u\in L^1_{loc}(I)$ is enough)

$\uparrow$ $u\in L^1(a+\varepsilon,b-\varepsilon)$ $\forall\varepsilon>0$

Let $v(x)\in L^p(I)$ ($v\in L^1_{loc}$ is enough)

We will say that v is the weak derivative of u if

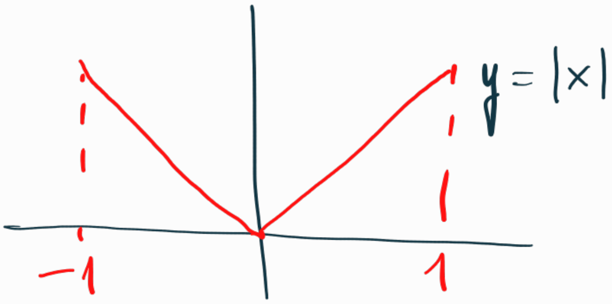
$$\int_a^b u(x)\,\varphi'(x)\,dx=-\int_a^b v(x)\,\varphi(x)\,dx \quad \forall\varphi\in C^1_c(I)$$

DEF Let $u\in L^p(I)$ s.t. it admits weak derivative $v\in L^p(I)$. Then we say that $u\in W^{1,p}(I)$.

Remark If $u(x) \in C^1([a,b])$, then $(**)$ holds, then $u'(x)$ (the classical derivative) is also the weak derivative, so

$$u \in W^{1,p}(I) \qquad \forall\, p \in [1,\infty]$$

Examples Take $I=(-1,1)$, $u(x)=|x|$



I want to show that $u \in W^{1,p}(I)$ $\forall\, p \in [1,+\infty]$. and its weak derivative is

$$V(x) = \begin{cases} 1 & x \in (0,1) \\ -1 & x \in (-1,0) \end{cases} \Bigg\} = \text{sign } x$$

I want to show tha

$$\int_{-1}^{1} |x|\,\varphi'(x)\,dx = -\int_{-1}^{1} \text{sign } x\ \varphi(x)\,dx \qquad \forall \varphi \in C_c^1(-1,1)$$

$$\|$$

$$-\int_{-1}^{0} x\,\varphi'(x)\,dx + \int_{0}^{1} x\,\varphi'(x)\,dx$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\int_{-1}^{0} 1\,\varphi(x)\,dx + \cancel{x\varphi(x)\Big|_{-1}^{0}} \qquad -\int_{0}^{1} 1\,\varphi(x)\,dx$$

$$= \int_{-1}^{1} (-\text{sign } x)\,\varphi(x)\,dx$$

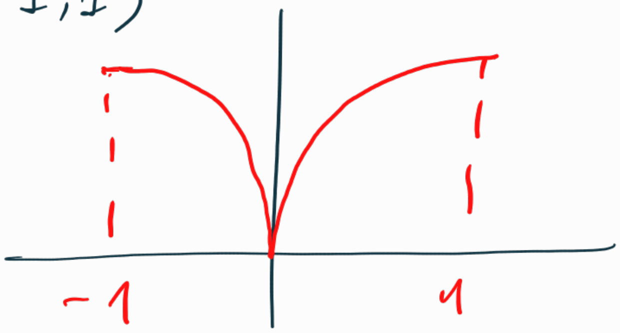
Moreover, $u(x), v(x) \in L^{\infty}(I) \Rightarrow$

$$\begin{array}{c} u \in W^{1,\infty}(I) \\ \cap\!| \\ W^{1,p}(I) \\ \forall p \in [1,\infty) \end{array}$$

Example 2.

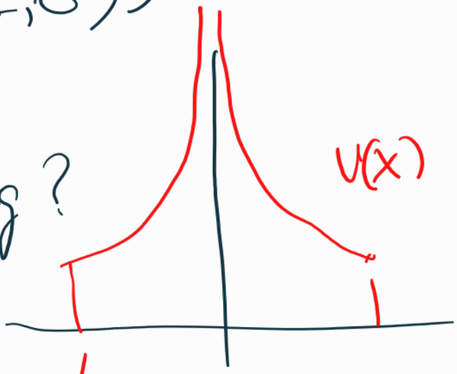
$I = (-1,1)$

$u(x) = \sqrt{|x|}$



I want to show (later) that its weak derivative is

$$v(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} & \text{if } x \in (0,1) \\ -\dfrac{1}{2\sqrt{-x}} & x \in (-1,0) \end{cases} = \frac{\operatorname{sign} x}{2\sqrt{|x|}}$$



To which Sobolev space does u belong?

$u \in L^p(I) \quad \forall p \in [1, +\infty]$.

$v \in L^p(I) \iff p < 2$

$$\int_{-1}^{1} \left| \frac{1}{2\sqrt{|x|}} \right|^p = \frac{2}{2^p} \int_0^1 \frac{1}{x^{p/2}}\, dx < \infty \underset{p/2 < 1}{\Updownarrow}$$

$\Longrightarrow u \in W^{1,p}(I) \quad \forall\, 1 \leq p < 2$

I want to prove that v is the weak derivative

$$\int_I u\varphi'\,dx = -\int_I v\varphi\,dx \qquad \forall\varphi\in C_c^1(I)$$

$$\int_{-1}^{1}\sqrt{|x|}\,\varphi'(x)dx = -\int_{-1}^{1}\frac{\operatorname{sign} x}{2\sqrt{|x|}}\varphi(x)dx$$

$$\parallel$$

$$\int_{-1}^{0}\sqrt{-x}\,\varphi'(x)dx + \underbrace{\boxed{\int_{0}^{1}\sqrt{x}\,\varphi'(x)dx}}_{} = -\int_0^1\frac{\varphi(x)}{2\sqrt{x}}dx + \text{☺}$$

$$\parallel$$

$$\underbrace{\int_0^{\varepsilon}\sqrt{x}\,\varphi'(x)dx}_{\text{this goes to zero as } \varepsilon\to 0 \ \text{☺}} + \underbrace{\int_{\varepsilon}^{1}\sqrt{x}\,\varphi'(x)dx}_{}$$

$$\parallel$$

$$-\int_{\varepsilon}^{1}\frac{1}{2\sqrt{x}}\varphi(x)dx + $$

$$+\ \underbrace{\sqrt{x}\,\varphi(x)\Big|_{\varepsilon}^{1}}_{\substack{\parallel \\ -\sqrt{\varepsilon}\,\varphi(\varepsilon) \\ \parallel \\ \text{☺}}}$$

$$= -\int_{\varepsilon}^{1}\frac{1}{2\sqrt{x}}\varphi(x)dx + \text{☺}$$

$$\parallel$$

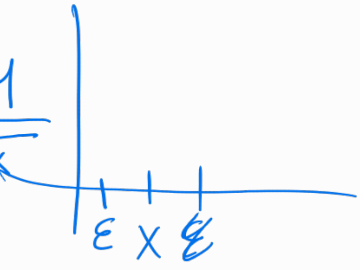
$$= -\underbrace{\int_0^1\frac{\varphi(x)}{2\sqrt{x}}dx}_{\text{finite.}} + \underbrace{\int_0^{\varepsilon}\frac{\varphi(x)}{2\sqrt{x}}dx}_{\parallel \ \text{☺}} = \int$$

$$\int_0^\varepsilon \underset{\substack{\cap\\ L^1}}{\boxed{\frac{\varphi(x)}{2\sqrt{x}}}}\,dx \xrightarrow{\varepsilon\to 0^+} 0$$

$$\shortparallel$$

$$\int_I \underbrace{\frac{\varphi(x)}{2\sqrt{x}}\,\chi_{(0,\varepsilon)}}_{\substack{\downarrow \text{ a.e. as } \varepsilon\to 0\\ 0}}\,dx$$

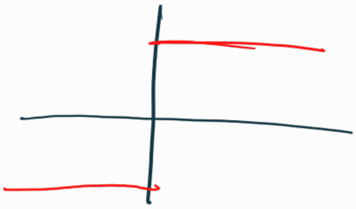
and $\left|\frac{\varphi(x)}{2\sqrt{x}}\chi_{(0,\varepsilon)}\right| \le \underset{\substack{\cap\\ L^1}}{\frac{M}{\sqrt{x}}}$



Example 3

I will show that the function

$$u(x) = \operatorname{sign} x = \begin{cases} -1 & \text{in } (-1,0) \\ 1 & \text{in } (0,1) \end{cases}$$



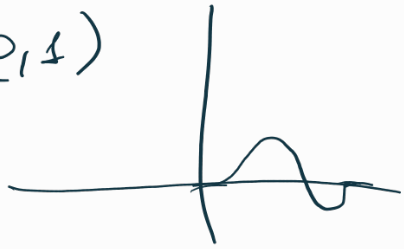
does not belong to any sobolev space. Actually, it does not have a weak derivative

Suppose, by contradiction, there exist $v \in L^1(-1,1)$ weak derivative of u.

Then it must be

$$\int_{-1}^1 \underset{\substack{\shortparallel\\ \text{sign } x}}{u(x)}\,\varphi'(x)\,dx = -\int_{-1}^1 v(x)\varphi(x)\,dx \qquad \forall \varphi \in C_c^1(-1,1)$$

In particular, I can take $\varphi \in C_c^1(0,1)$



$$\int_0^1 1\,\varphi'(x)\,dx = -\int_0^1 v(x)\varphi(x)\,dx$$

$$\|$$

$$\varphi(1)-\varphi(0)=0$$

$$\Rightarrow \int_0^1 v(x)\varphi(x)\,dx = 0 \qquad \forall \varphi \in C_c^1(0,1)$$

LEMMA Fundamental Lemma of Calculus of Variations

If $v \in L^1(a,b)$ s.t. $\int_a^b v(x)\varphi(x)\,dx = 0 \quad \forall \varphi \in C_c^1(a,b$

$\Longrightarrow V = 0$ a.e on a,b

$\Rightarrow V \equiv 0$ a.e on $(0,1)$

Reasoning in the same way, I can prove that $V \equiv 0$ a.e. on $(-1,0)$

$\Longrightarrow V \equiv 0$ a.e. in $(-1,1)$

Now I take $\varphi \in C_c^1(-1,1)$

$$\int_{-1}^1 \operatorname{sign} x \; \varphi'(x)\,dx = -\int_{-1}^1 v(x)\varphi(x)\,dx = 0$$

$$\|$$

$$-\int_{-1}^0 1\,\varphi'(x)\,dx + \int_0^1 \varphi'(x)\,dx =$$

$$= \cancel{\varphi(-1)} - \varphi(0) + \cancel{\varphi(1)} - \varphi(0) = -2\varphi(0)$$

Absurd because $\varphi(0)$ can be nonzero.

So, we have proven that the weak derivative of sign x is not an L^1 function.

Actually, its weak derivative is not a function, it is a measure
the weak derivative of sign x is $2\delta_0(x)$

Actually, we have seen that

$$\int_{-1}^{1} \operatorname{sign} x \, \varphi'(x)\,dx = - \quad 2\varphi(0) = - \underbrace{\int 2\delta_0 \varphi}_{\substack{\parallel \\ -\,2\varphi(0)}}$$

$$(\operatorname{sign} x) = 2\delta_0$$