H spazio con prodotto scalare.

E' uno spazio vettoriale dotato di un prodotto scalare, verificante

i) $(x,y) = (y,x) \quad \forall x,y \in H$

ii) bilineare (cioè lineare rispetto a ognuno dei due vettori

iii) $(x,x) \geq 0 \quad \forall x \in H$

iv) $(x,x) = 0 \iff x = 0$

OSS uno spazio con prodotto scalare è anche normato, con la norma $\|x\| = \sqrt{(x,x)}$

DEF Uno spazio con prodotto scalare si dice spazio di Hilbert se è completo come spazio metrico

ossia: $\{x_n\}$ di Cauchy $\Rightarrow$ x_n converge a un certo x

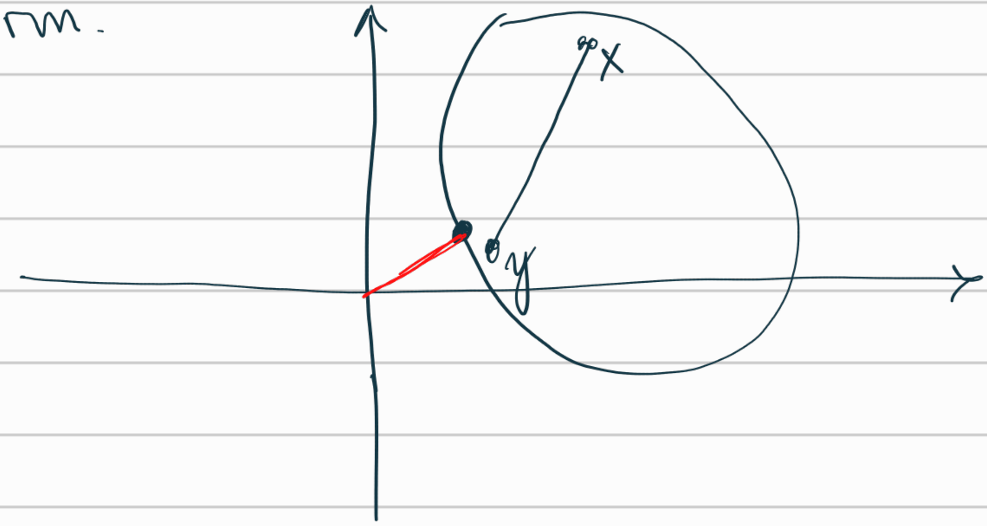
$\Updownarrow$ def

$\forall \varepsilon > 0 \ \exists \bar{n}$ t.c. $\|x_n - x_m\| < \varepsilon \ \forall n,m \geq \bar{n}$

THEOREM. H Hilbert space

Let $E \subseteq H$ be a closed and convex subset.

Then E admits a unique point x of minimal norm.



E convex means:

$$\forall x, y \in E \quad \forall t \in (0,1) \qquad tx + (1-t)y \in E$$

E closed means.

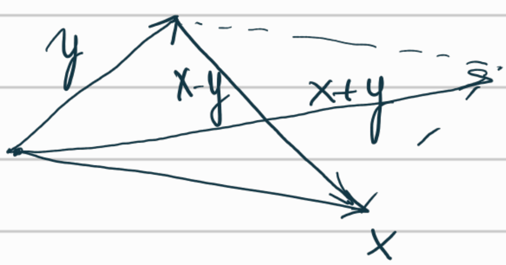
$$\left.\begin{array}{l} \text{If } x_n \in E \\ x_n \to x \end{array}\right| \Rightarrow x \in E$$

Proof of Thm.

LEMMA Parallelogram identity.

$\forall x, y \in H$ space with scalar product

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$



Proof of lemma. It is enough to expand all norms, for instance

$$\|x+y\|^2 = (x+y, x+y) = \underset{\|x\|^2}{\underset{\shortparallel}{(x,x)}} + \underset{\|y\|^2}{\underset{\shortparallel}{(y,y)}} + 2(x,y)$$

Proof of theorem.

Let $\delta = \inf \{\|x\|,\ x \in E\}$

We want to check that $\exists!\ x \in E$ s.t. $\|x\| = \delta$

Consequence of the parallelogram identity

$\forall x, y \in E$

$$\|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \underbrace{\Big\|\overset{E \text{ by convexity}}{\overset{\in}{\frac{x+y}{2}}}\Big\|^2}_{\geq \delta^2} 4 \leq$$

$$\leq 2\|x\|^2 + 2\|y\|^2 - 4\delta^2$$

Uniqueness of the point x.

Assume that $x, y \in E$ satisfy $\|x\| = \|y\| = \delta$

$$\|x-y\|^2 \leq 2\underset{\delta^2}{\underset{\shortparallel}{\|x\|^2}} + 2\underset{\delta^2}{\underset{\shortparallel}{\|y\|^2}} - 4\delta^2 = 0 \Rightarrow x = y.$$

Existence Let $\{x_n\}$ be a "minimizing sequence", i.e.

$x_n \in E, \qquad \|x_n\| \to \delta.$

Then this sequence is Cauchy.

$$\|x_n - x_m\|^2 \leq 2\underbrace{\|x_n\|^2}_{\searrow \delta^2} + 2\underbrace{\|x_m\|^2}_{\searrow \delta^2} - 4\delta^2 \xrightarrow{\text{for } n,m \text{ large}} 0$$

H is complete, $x_n \to x \in H$
E is closed $\Rightarrow$ $x \in E$.
If we show that $\|\cdot\|$ is a continuous function

$$\|x_n\| \longrightarrow \delta = \inf\{\|y\|,\ y \in E\}.$$
$$\downarrow$$
$$\|x\| \qquad \Rightarrow \quad \|x\| = \delta \quad \text{as required.}$$

The norm is continuous, indeed, by the triangle inequality

$$\|x\| = \|x-y+y\| \le \|x-y\| + \|y\|$$

$$\Rightarrow \quad \|x\| - \|y\| \le \|x-y\|$$

exchange x and y

$$\|y\| - \|x\| \le \|x-y\|$$

$$x_n \to x \quad \Downarrow \quad \|x_n\| \to \|x\|$$

$$\big|\|x\| - \|y\|\big| \le \|x-y\| \Rightarrow \|x\| \text{ is a continuous function on } H.$$

Orthogonal projections.

Def $x, y \in H$. We say that they are orthogonal if $(x, y) = 0$, and we will write $x \perp y$

Def. Let M be a vector subspace of H.
We define the orthogonal complement $M^\perp$ of M as

$$M^\perp = \{y \in H \text{ s.t. } \ y \perp x \quad \forall x \in M\}.$$

Rem $\quad M^\perp$ is also a subspace.

$$\left.\begin{array}{l} y \in M^\perp \\ z \in M^\perp \end{array}\right| \overset{?}{\longrightarrow} \quad y+z \in M^\perp$$

Let $x \in M$. $\Rightarrow \quad (y+z, x) = \underset{\overset{\|}{0}}{(y,x)} + \underset{\overset{\|}{0}}{(z,x)} = 0$

Similarly

$$\left.\begin{array}{l} y \in M^\perp \\ \lambda \in \mathbb{R} \end{array}\right| \Rightarrow \quad \lambda y \in M^\perp$$

So $\quad M^\perp$ is a subspace.

It is also a <u>closed</u> subspace.

$$\left.\begin{array}{l} y_n \in M^\perp \\ y_n \to y \end{array}\right| \overset{?}{\Longrightarrow} \quad y \in M^\perp$$

let $x \in M$, $0 = (y_n, x) \overset{?}{\longrightarrow} (y, x) = 0$

Infact

$$|(y_n, x) - (y, x)| = |(y_n - y, x)| \overset{\text{Cauchy-Schwarz}}{\leq}$$

$$\leq \underbrace{\|y_n - y\|}_{\downarrow \, 0} \|x\|$$

So, we have proven that the scalar product is continuous with respect to each of the two variables.

<u>Remark</u> Not all subspaces of a Hilbert space are closed.

For instance:

$$C([a,b]) \subseteq L^2(a,b)$$

$\uparrow$ subspace, but it is not closed.

<u>THEOREM of ORTHOGONAL PROJECTIONS</u>

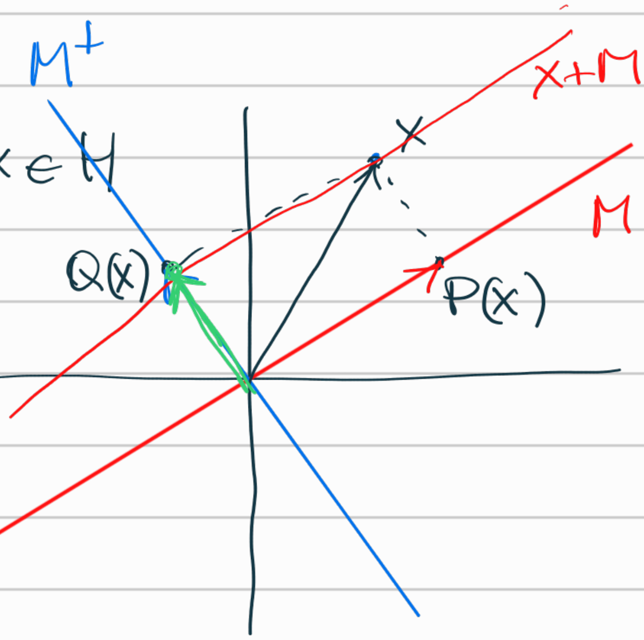
H Hilbert space, $M \subseteq H$ closed subspace

Then there exists a unique pair of functions

$$P : H \longrightarrow M$$

$$Q : H \longrightarrow M^\perp$$

s.t. $\quad x = P(x) + Q(x) \quad \forall x \in H$



<u>Moreover:</u>

1) If $x \in M \Rightarrow P(x) = x, \, Q(x) = 0$

If $x \in M^\perp \Rightarrow P(x) = 0, \, Q(x) = x$

2) $\|x - P(x)\| = \min\{\|x-y\| : y \in M\}$

3) $\|x\|^2 = \|P(x)\|^2 + \|Q(x)\|^2$ (Pythagoras's theorem).

4) P, Q are linear.

Proof. Let $x \in H$. We consider the set

$$x + M = \{x+y : y \in M\}$$ is convex and closed.

So, by the previous theorem,
$\exists!$ element of $x+M$ of minimal norm.
This I will call $Q(x)$, and I define

$$P(x) = x - Q(x).$$

I must prove that $P(x) \in M$
$Q(x) \in M^\perp$

$$Q(x) \in x+M \Rightarrow Q(x) = x+y \quad , \quad y \in M$$

$$P(x) = x - Q(x) = \cancel{x} - (\cancel{x}+y) = -y \in M.$$

$$(Q(x), y) \overset{?}{=} 0 \qquad \forall y \in M^\perp$$

I can always assume $\|y\| = 1$, otherwise replace y by $\frac{y}{\|y\|}$

Let $z = Q(x)$. By minimality

$$\cancel{\|z\|^2} \leq \|z+\alpha y\|^2 = (z+\alpha y, z+\alpha y) =$$

$$= \cancel{\|z\|^2} + \alpha^2 \|y\|^2 + 2\alpha(z,y)$$

$$\alpha^2\|y\|^2 + 2\alpha(z,y) \geq 0 \qquad \forall \alpha \in \mathbb{R}$$

$$\Rightarrow 0 \geq \Delta = 4(z,y)^2 \Rightarrow (z,y) = 0 \Rightarrow Q(x) \in M^{\perp}$$

This decomposition is unique: indeed

Assume that $x = \underset{\substack{\cap \\ M}}{P(x)} + \underset{\substack{\cap \\ M^{\perp}}}{Q(x)} = \underset{\substack{\cap \\ M}}{x_1} + \underset{\substack{\cap \\ M^{\perp}}}{x_2}$

$$\Rightarrow \underset{\substack{\cap \\ M}}{P(x) - x_1} = \underset{\substack{\cap \\ M^{\perp}}}{x_2 - Q(x)} \Longrightarrow \text{since } M \cap M^{\perp} = \{0\}$$

$$P(x) = x_1 \,, \quad Q(x) = x_2 \Rightarrow \text{uniqueness.}$$

Now we show 4) (Linearity of P and Q).

let $x, y \in H$, $\alpha, \beta \in \mathbb{R}$.

We want to show that

$$\underset{Q}{P}(\alpha x + \beta y) = \alpha \underset{Q}{P}(x) + \beta \underset{Q}{P}(y)$$

$$x = P(x) + Q(x)$$
$$y = P(y) + Q(y)$$

$$\alpha x + \beta y = \alpha P(x) + \alpha Q(x) + \beta P(y) + \beta Q(y)$$

$$\|$$

$$\alpha x + \beta y = P(\alpha x+\beta y) + Q(\alpha x+\beta y)$$

$$\Rightarrow \overbrace{\alpha P(x) + \beta P(y) - P(\alpha x+\beta y)}^{\in M} =$$

$$= \underbrace{Q(\alpha x+\beta y) - \alpha Q(x) - \beta Q(y)}_{\in M^{\perp}}$$

so both r.h.s. and l.h.s. must be zero $\Rightarrow$ linearity.

let us prove 3) Pythagora's identity

$$\|x\|^2 = (P(x)+Q(x), P(x)+Q(x)) =$$

$$= \|P(x)\|^2 + \|Q(x)\|^2 + 2 \underbrace{(P(x), Q(x))}_{= 0}$$

2) is trivial, since

$$\|x - P(x)\| = \min \{ \|x+y\|, \ y \in M \}$$

I write $y = -\tilde{y}$ with $\tilde{y} \in M$

$$= \min \{ \|x - \tilde{y}\|, \ \tilde{y} \in M \}$$

Fourier series in Hilbert spaces

V vector space, $u_1, u_2, \dots u_k \in V$

DEF $u_1, u_2, \dots u_k$ are linearly independent if

$$u = c_1 u_1 + c_2 u_2 + \dots + c_k u_k = 0 \quad (c_1, c_2, \dots c_k \in \mathbb{R})$$

$$\Downarrow$$

$$c_1 = c_2 = \dots = c_k = 0$$

Now, if H is a vectorspace with scalar product.

DEF $u_1, u_2, \dots u_k$ are orthonormal if

$$(u_i, u_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Remark

$u_1, \dots u_k$ orthonormal $\Longrightarrow$ $u_1, \dots u_k$ linearly independent.

Assume that $u_1, \dots u_k$ are orthonormal.
If

$$0 = c_1 u_1 + c_2 u_2 + \dots + c_k u_k = \sum_{i=1}^{k} c_i u_i$$

$$0 = (0, 0) = \left(\sum_{i=1}^{k} c_i u_i , \sum_{j=1}^{k} c_j u_j \right) = \text{(bilinearity)}$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} c_i c_j (u_i, u_j) = \sum_{i=1}^{k} c_i^2 \cdot 1$$

$$\Rightarrow c_i = 0 \qquad \forall i = 1 \dots k.$$

PROP $\{u_1, \dots u_k\}$ orthonormal set of H.

If $x = c_1 u_1 + \dots + c_k u_k = \sum_{i=1}^{k} c_i u_i$, then

1) $c_i = (x, u_i)$ ← Fourier coefficients $\qquad \forall i = 1 \dots k$

2) $\|x\|^2 = \sum_{i=1}^{k} c_i^2$ Pythagoras' theorem.

Proof of 1) $(x, u_i) = \left(\sum_{j=1}^{k} c_j u_j, u_i \right) =$

$$= \sum_{j=1}^{k} c_j (u_j, u_i) = c_i$$

Proof of 2)

$$\|x\|^2 = (x, x) = \left(\sum_{i=1}^{k} c_i u_i, \sum_{j=1}^{k} c_j u_j \right) = \text{as before}$$

$$= \sum_{i=1}^{k} c_i^2 \qquad \square$$

Let $\{u_1, \dots u_k\}$ be an orthonormal set in H Hilbert space

I want to approximate a generic point of H, say $x \in H$, with linear combinations of $\{u_1, \dots u_k\}$.

I need to find constant $c_1, \dots c_k$ such that the linear combination $\sum_{i=1}^{k} c_i u_i$ best approximates x, i.e. such that

$$\left\| x - \sum_{i=1}^{k} c_i u_i \right\| \text{ is minimal.}$$

So we want to use orthogonal projections on the subspace

$$M = \operatorname{span}\{u_1, \dots u_k\} = \left\{ \sum_{i=1}^{k} c_i u_i, \ c_i \in \mathbb{R} \right\}$$

It is clear that M is a subspace

We must show that M is <u>closed</u>.

So assume that $\{x^{(n)}\} \in M$, $x^{(n)} \to x$.

We must show that $x \in M$

$\{x^{(n)}\}$ converges $\Rightarrow$ $\{x^{(n)}\}$ is a Cauchy sequence

$\Rightarrow \forall \varepsilon > 0 \ \exists \bar{n}$ s.t. $\forall n, m > \bar{n}$ $\|x^{(m)} - x^{(n)}\| < \varepsilon$

$$\varepsilon^2 > \|\underbrace{x^{(m)}}_{= \sum_{i=1}^{k} c_i^{(m)} u_i} - \underbrace{x^{(n)}}_{= \sum c_i^{(n)} u_i}\|^2 = \sum_{i=1}^{k} (c_i^{(m)} - c_i^{(n)})^2 \geq (c_j^{(m)} - c_j^{(n)})^2 \quad \forall j$$

$\Rightarrow \{c_j^{(n)}\}$ is a Cauchy sequence in $\mathbb{R}$

$\Rightarrow c_j^{(n)} \to c_j \quad \forall j = 1 \dots k$

Now we set $\quad y = \sum_{j=1}^{k} c_j u_j$

$$\|x^{(n)} - y\|^2 = \sum_{j=1}^{k} (c_j^{(n)} - c_j)^2 \xrightarrow{n \to +\infty} 0$$

$$\left.\begin{array}{l} x^{(n)} \to y \\ x^{(n)} \to x \end{array}\right| \Rightarrow y = x \qquad \blacksquare$$

Therefore $M = \text{span}\{u_1, \dots, u_k\}$ is a closed subspace of a Hilbert space.

$\Rightarrow$ I can take orthogonal projections

$\forall x \in H \quad \exists!\ Px \in M,\ Qx \in M^{\perp}$ s.t.

$x = \underbrace{Px}_{x_0} + Qx$. Moreover

$$\|x - x_0\| = \min_{y \in M} \|x - y\| = \text{dist}(x, M)$$

$$x - x_0 \in M^{\perp}$$

$$\begin{array}{c} \| \\ x - Px \\ \| \\ Qx \end{array}$$

$$\Rightarrow \quad x - x_0 \perp u_i \qquad \forall i = 1 \dots k$$

4

$$(x - x_0, u_i) = 0$$

$$(x, u_i) = (x_0, u_i) = c_i$$

$$\Rightarrow \quad x_0 = \sum_{i=1}^{k} c_i u_i = \sum_{i=1}^{k} (x, u_i) u_i$$

$$c_i = (x_0, u_i) = (x, u_i)$$

<u>Summary</u> If $\{u_1, \dots u_k\}$ is an orthonormal set in H Hilbert space, and $x \in H$, the $\boxed{\text{best approximation}}$ of x by al element of $M = \text{span}[u_1, \dots u_k]$ is the following

$$x_0 = \sum_{i=1}^{k} (x, u_i) u_i$$

It means that the norm $\|x - x_0\|$ is minimal.

In formulas.

1) $$\left\| x - \sum_{i=1}^{k} (x, u_i) u_i \right\| \leq \left\| x - \sum_{i=1}^{k} \lambda_i u_i \right\|$$

$$\forall \lambda_1 \dots \lambda_k \in \mathbb{R}$$

2) $\sum_{i=1}^{k} (x, u_i)^2 \leq \|x\|^2$ (Bessel inequality).

Proof

$$\|x\|^2 = \|P(x)\|^2 + \overbrace{\|Q(x)\|^2}^{\geq 0} \geq \|P(x)\|^2 =$$

$$= \left\| \sum_{i=1}^{k} (x, u_i) u_i \right\|^2$$

$$\parallel$$

$$\sum_{i=1}^{k} (x, u_i)^2$$

Application $H = L^2(-\pi, \pi)$ Hilbert space.

$$u_0(t) = \frac{\sqrt{2}}{2}$$

$$u_k(t) = \cos(kt) \qquad k = 1, 2 \ldots$$

$$v_k(t) = \sin(kt) \qquad k = 1, 2 \ldots$$

Let us check that these functions are orthonormal

On $L^2(-\pi, \pi)$ we use the scalar product

$$(f, g) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) g(t) dt$$

$$\|u_0\|^2 = (u_0, u_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} dt = 1.$$

$$\|u_k\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(kt)\,dt = \qquad k=1,2\ldots$$

$$\cos^2 s = \frac{1+\cos(2s)}{2}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2} + \underbrace{\frac{1}{2}\cos(2kt)}_{\text{the integral is zero}}\right) dt = 1.$$

$$\sin^2 s = \frac{1-\cos(2s)}{2}$$

$$\|v_k\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(kt)\,dt = 1$$

$$(u_0, u_k) = \qquad k=1,2\ldots$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sqrt{2}}{2}\cos(kt)\,dt = 0$$

$$(u_0, v_k) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sqrt{2}}{2}\sin(kt) = 0$$

$$(u_k, u_h) = \qquad k \neq h$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(kt)\cos(ht)\,dt =$$

$$\int_{-\pi}^{\pi} \cos(kt)\cos(ht)\,dt = \cancel{\frac{1}{h}\cos(kt)\sin(ht)\Big|_{-\pi}^{\pi}} -$$

$$- \frac{k}{h}\int_{-\pi}^{\pi} \sin(kt)\sin(ht)\,dt =$$

$$= \frac{k}{h^2} \cancel{\sin(kt)\cos(ht)\Big|_{-\pi}^{\pi}} + \frac{k^2}{h^2}\int_{-\pi}^{\pi} \cos(kt)\cos(ht)\,dt$$

$$\Rightarrow \underbrace{\left(1 - \frac{k^2}{h^2}\right)}_{\substack{\neq \\ 0}} \int_{-\pi}^{\pi} \cos(kt)\cos(ht)\,dt = 0$$

and so on

Fix n, and consider a trigonometric polynomial of the form.

$$a_0 u_0 + \sum_{k=1}^{n} \big(a_k u_k(t) + b_k v_k(t)\big)$$

For a fixed function f in $L^2(-\pi,\pi)$
i want to choose a_0, a_k, b_k such that the distance from f to the trig. polynomial is minimal
By the theory just exposed, we must take

$$a_k = (f, u_k) \quad , \quad b_k = (f, v_k)$$