Spazi L^p

Important cases: $p = 1, 2, +\infty$

$\Omega \subseteq \mathbb{R}^N$ misurabile.

References:

Royden - Real Analysis

Rudin - Real and Complex Analysis

Brezis - Functional Analysis

DEF $\boxed{1 \leq p < \infty}$

$$L^p(\Omega) = \left\{ f : \Omega \to \overline{\mathbb{R}} \text{ Lebesgue measurable s.t. } \int_\Omega |f(x)|^p \, dx < +\infty \right\}$$

after identifying functions which are equal a.e. in Ω.

$$\|f\|_p = \left[\int_\Omega |f(x)|^p \, dx \right]^{1/p} \quad \text{is a norm.}$$

DEF $\boxed{p = \infty}$

$$L^\infty(\Omega) = \left\{ f(x) : \Omega \to \overline{\mathbb{R}} \text{ Lebesgue measurable such that } |f(x)| \leq M \text{ a.e. in } \Omega \text{ for some } M \geq 0 \right\}$$

$$= \left\{ f(x) : \Omega \to \overline{\mathbb{R}} \text{ measurable and essentially bounded} \right\}$$

After (again) identifying functions which are equal a.e.

$$\|f\|_\infty = \min \left\{ M \geq 0 : |f(x)| \leq M \text{ a.e. in } \Omega \right\}$$

$$\Rightarrow \quad |f(x)| \leq \|f\|_\infty \quad \text{a.e. in } \Omega$$

Important inequalities:

1) Young's inequality.

Hölder conjugate exponents

If $p,q \in (1,+\infty)$ verify $\frac{1}{p}+\frac{1}{q}=1$, then

$q = \frac{p}{p-1}$

$$\boxed{xy \leq \frac{x^p}{p} + \frac{y^q}{q} \quad \forall\, x,y \geq 0.}$$

2) Hölder's inequality

If $p,q \in [1,+\infty]$ satisfy $\frac{1}{p}+\frac{1}{q}=1$,

if $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$, and

$$\boxed{\int_\Omega |fg|\,dx \leq \|f\|_p \, \|g\|_q}$$

Triangle inequality (Minkowski)

$p \in [1, +\infty]$; $f, g \in L^p(\Omega) \implies$

$$\|f+g\|_p \le \|f\|_p + \|g\|_p \quad (\implies \|\cdot\|_p \text{ is a norm})$$

Proof $\boxed{p=\infty} \implies$ already done.

$\boxed{p=1} \implies$ easy.

triangle inequality for numbers

$$\|f+g\|_1 = \int_\Omega |f(x)+g(x)|\,dx \le \int_\Omega (|f(x)| + |g(x)|)\,dx = $$

$$= \|f\|_1 + \|g\|_1$$

$\boxed{1 < p < \infty}$

$$\|f+g\|_p^p = \int_\Omega |f(x)+g(x)|^p\,dx = \int_\Omega |f(x)+g(x)|^{p-1} \underbrace{|f(x)+g(x)|}_{\le\, |f(x)|+|g(x)|}\,dx$$

$$\le \int_\Omega \underbrace{|f(x)+g(x)|^{p-1}}_{\in\, L^{p/(p-1)=q}} \underbrace{|f(x)|}_{\in\, L^p}\,dx + \int_\Omega |f+g|^{p-1}|g|\,dx$$

Hölder

$$\le \underbrace{\left[\int |f+g|^{\frac{\cancel{(p-1)}p}{\cancel{p-1}}}\right]^{\frac{p-1}{p}}}_{=\, \|f+g\|_p^{p-1}} \|f\|_p + \|f+g\|_p^{p-1}\|g\|_p =$$

$$= \|f+g\|_p^{p-1}\left(\|f\|_p + \|g\|_p\right)$$

$\Rightarrow$ dividing by $\|f+g\|_p^{p-1}$, we obtain the inequality $\square$

Rem If $m(\Omega)<\infty$, then the L^p spaces are all nested into one another, in the sense that, if $1\leq p<r\leq\infty$, then $L^r(\Omega)\subset L^p(\Omega)$.

<u>Proof</u> if $r<\infty$

$f\in L^r(\Omega)$

$$\int_\Omega |f|^p dx = \int_\Omega |f|^p\cdot 1\, dx \leq$$ Hölder's inequality with exponents $\left(\frac{r}{p}, \frac{r}{r-p}\right)$

$$\leq \underbrace{\left[\int_\Omega |f|^{\cancel{p}\frac{r}{\cancel{p}}}\right]^{\frac{p}{r}}}_{\|f\|_r^p} \underbrace{\left[\int_\Omega 1\right]^{\frac{r-p}{r}}}_{m(\Omega)^{\frac{r-p}{r}}}$$

$$\frac{r/p}{\frac{r}{p}-1} = \frac{r}{r-p}$$

So we obtain $\|f\|_p \leq c(m(\Omega),p,r)\,\|f\|_r$

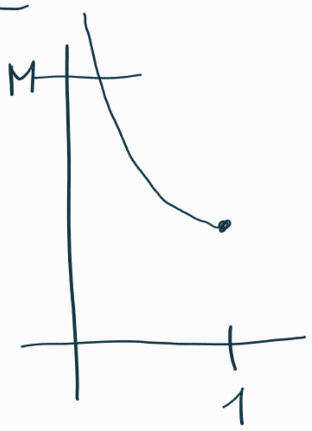
<u>Rem</u> It follows from the last inequality that:

If $f_n\to f$ in $L^r(\Omega)$, with $r>p$, $m(\Omega)<\infty$

then $f_n\to f$ in $L^p(\Omega)$

Examples $\quad N=1 \quad \Omega=(0,1) \quad f(x)=\frac{1}{\sqrt{x}} \overset{?}{\in} L^p(0,1)$

for what p?



$f(x) \notin L^\infty(0,1)$

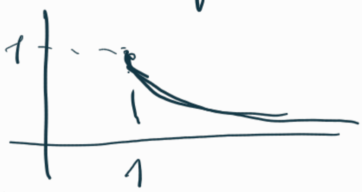
$$\int_0^1 |f(x)|^p\,dx = \int_0^1 \frac{1}{x^{\frac{1}{2}\cdot p}}\,dx < \infty \iff \frac{p}{2} < 1$$

$$\Updownarrow$$

$$\boxed{p<2}$$

$N=1 \quad \Omega=(1,+\infty) \quad f(x)=\frac{1}{\sqrt{x}} \in L^p(1,+\infty)$

for which p?

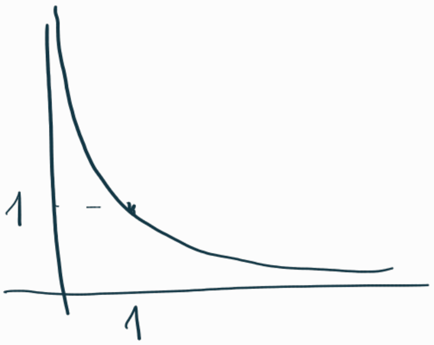


$$\int_1^{+\infty} |f(x)|^p\,dx = \int_1^{+\infty} \frac{dx}{x^{p/2}} < \infty \iff \frac{p}{2} > 1 \iff p>2$$

and also $p=\infty$

$N=1, \quad \Omega=(0,+\infty), \quad f(x)=\frac{1}{\sqrt{x}} \in L^p(0,+\infty)$

for which p?

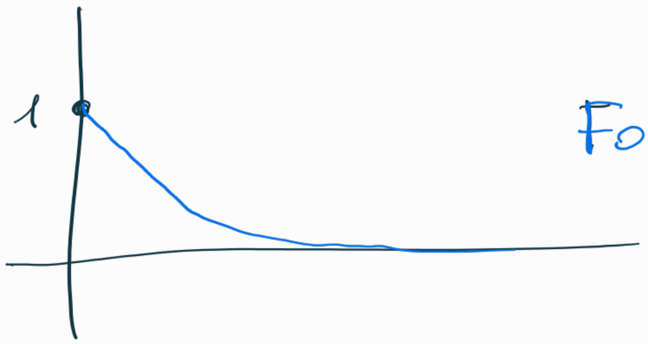


For no $p \in [1,+\infty)$

- $N=1$ $\quad \Omega=(0,+\infty)$ $\quad f(x)=e^{-x}\in L^p(0,+\infty)$

for which p?

For every $p\in[1,+\infty]$



$f\in L^\infty(0,+\infty)$

If $1\le p<\infty$

$$\int_\Omega |f(x)|^p\,dx=\int_0^{+\infty}e^{-px}\,dx=-\frac{1}{p}e^{-px}\Big|_0^{+\infty}=$$

$$=\frac{1}{p}\Big(1-\underbrace{e^{-p(+\infty)}}_{=0}\Big)=\frac{1}{p}$$

$N=3$ $\quad f(x,y,z)=\dfrac{1}{x^2+y^2+z^2}$; $\Omega=B_1(0)=\{(x,y,z):x^2+y^2+z^2<1\}$

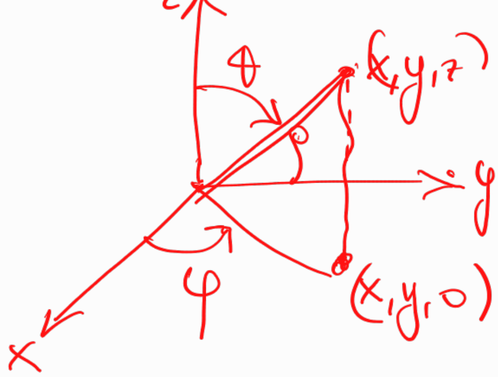
$f\in L^p(B_1(0))$ for which p?

$f\notin L^\infty(B_1(0))$

$1\le p<\infty$

$$\iiint_{B_1(0)}\frac{dx\,dy\,dz}{(\underbrace{x^2+y^2+z^2}_{\rho^2})^p}=$$

Spherical coordinates.



$$\begin{cases}x=\rho\sin\theta\cos\varphi\\ y=\rho\sin\theta\sin\varphi\\ z=\rho\cos\theta\end{cases}$$

$dx\,dy\,dz=\rho^2\sin\theta\quad d\rho\,d\theta\,d\varphi$

$$= \int_0^{\pi} d\theta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \int_0^1 d\rho \, \frac{\rho^2 \sin\theta}{\rho^{2p}} =$$

$$= \underbrace{\int_0^{\pi} d\theta \sin\theta}_{2} \cdot 2\pi \underbrace{\int_0^1 \frac{d\rho}{\rho^{2p-2}}}_{\wedge \atop \infty} \iff 2p-2 < 1 \iff p < \frac{3}{2}$$

$$\boxed{1 \le p < \frac{3}{2}}$$

Why the norm L^∞ is denoted in this way?

Rem Suppose $m(\Omega) < \infty$.

$f \in L^\infty(\Omega) \Rightarrow f \in L^p \quad \forall p.$

I compute $\lim_{p \to +\infty} \|f\|_p = \|f\|_\infty$

Proof

$$\int_\Omega \underbrace{|f(x)|^p}_{\le \|f\|_\infty^p} dx \le \|f\|_\infty^p \, m(\Omega)$$

$$\Rightarrow \quad \le \|f\|_p \le \|f\|_\infty \underbrace{(m(\Omega))^{1/p}}_{\downarrow \atop 1} \xrightarrow[p \to +\infty]{} \|f\|_\infty$$

$$\le \left[\int_{\Omega_\varepsilon} \underbrace{|f(x)|^p}_{\ge (\|f\|_\infty - \varepsilon)^p} dx\right]^{1/p}$$

$$(\|f\|_\infty - \varepsilon)(m(\Omega_\varepsilon))^{1/p} \xrightarrow[p \to +\infty]{} \|f\|_\infty - \varepsilon$$

where $\Omega_\varepsilon = \{ x \in \Omega : |f(x)| > \|f\|_\infty - \varepsilon \}$ $\quad \varepsilon > 0$

this set has positive measure by def of $\|\cdot\|_\infty$

From these inequalities, since ε is arbitrary, it follows easily that

$$\lim_{p \to +\infty} \|f\|_p = \|f\|_\infty$$

$\square$

Theorem $\forall p \in [1, \infty]$, the normed space $L^p(\Omega)$ is complete. $\boxed{\text{Riesz-Fischer thm}}$

So $L^p(\Omega)$ is a $\underbrace{\text{normed space, complete}}_{\text{Banach space}}$

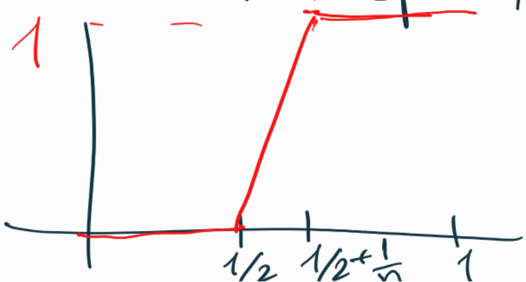
Proof of R-F theorem could be the subject of an end-of-course short seminar.

For instance, we know that if we take

$X = C^0([0,1])$ with the norm

$$\|f\|_2 = \left[\int_\Omega |f(x)|^2 dx \right]^{1/2}$$

$(X, \|\cdot\|_2)$ is a normed space, but it is not complete



Prop 1 (Consequence of the proof of R-F. theorem)

If $f_n \to f$ in $L^p(\Omega)$ $1 \leq p < \infty$,
then we can extract a subsequence $\{f_{k_n}\}$ such that

$f_{k_n} \to f$ a.e. in Ω.

How can you prove (in practice) that $f_n \to f$ in $L^p(\Omega)$?
A nice tool is the following proposition:

PROP. $1 \leq p < \infty$

Assume that

$$\left.\begin{array}{l} f_n(x) \to f(x) \text{ a.e. in } \Omega \\ |f_n(x)| \leq h(x) \in L^p(\Omega) \\ \qquad \forall n \in \mathbb{N} \\ \qquad \text{for a.e. } x \in \Omega \end{array}\right| \Longrightarrow f_n \to f \text{ in } L^p(\Omega)$$

Proof. I want to prove that

$$\int_\Omega |f_n(x) - f(x)|^p = \|f_n - f\|_p^p \xrightarrow[n \to +\infty]{} 0$$

Recall Lebesgue's dominated convergence theorem.

Let $h_n(x)$ be measurable functions on Ω s.t.

$h_n(x) \to h(x)$ a.e.

and $|h_n(x)| \leq K(x) \in L^1(\Omega)$ $\forall n$, a.e. $x \in \Omega$

Then $\int_\Omega h_n(x)\,dx \to \int_\Omega h(x)\,dx$

$$\int_\Omega \underbrace{|f_n(x) - f(x)|^p}_{\xrightarrow{\text{a.e.}}\ 0} \, dx \xrightarrow[\text{Lebesgue's thm}]{} \int_\Omega 0 = 0$$

triangle ineq.

$$|f_n(x) - f(x)|^p \le \left(|f_n(x)| + |f(x)|\right)^p \le \left(2\,h(x)\right)^p = 2^p\, h(x)^p$$

a.e. $\quad |f_n(x)| \le h(x) \qquad |f(x)| \le h(x) \qquad h(x)^p \in L^1$

THEOREM $1 \le p < \infty$, Ω open set in $\mathbb{R}^n$.

Let $f \in L^p(\Omega)$

Then, for every $\varepsilon > 0$, $\exists\, g_\varepsilon \in C_c(\Omega)$ s.t.

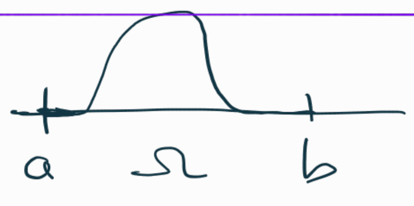
or $C^1_c(\Omega)$

$\|f - g_\varepsilon\|_p < \varepsilon$

or $C^2_c(\Omega)$

or $C^\infty_c(\Omega)$

where $C_c(\Omega)$ is the set of continuous functions on Ω with compact support.



this means that they are zero in a neighborhood of $\partial\Omega$ (the boundary)

this proof could also be a "final seminar".

The theorem is false in L^∞. If $f \in L^\infty(\Omega) \setminus C(\Omega)$ then it is not possible to approximate it in the L^∞ norm with continuous functions.

$L^2(\Omega)$ is "the best space of all L^p spaces", because we can define a scalar product.

Spaces with scalar product

Let H be a vector space.
A scalar product is a function from H^2 to $\mathbb{R}$.

$$(\cdot,\cdot) : H \times H \longrightarrow \mathbb{R}$$
$$x, y \longmapsto (x,y) = x \cdot y = \langle x, y \rangle$$

Satisfying.

1) $(x,y) = (y,x)$ In the case of complex vectorspace $(x,y) = \overline{(y,x)}$

2) $(x+y,z) = (x,z) + (y,z)$

3) $(\alpha x, z) = \alpha (x,z)$ $\forall \alpha \in \mathbb{R}$ $\}$ linearity w.r. to both variables

4) $(x,x) \geq 0$

5) $(x,x) = 0 \iff x = 0$

In a space with scalar product we can define a norm

$$\|x\| := \sqrt{(x,x)} \qquad \text{(see later)}$$

Examples

1) $\mathbb{R}$ $(x,y) = xy$ Hilbert

2) $\mathbb{C}$ $\quad (z,w) = z \cdot \overline{w}$ Hilbert

3) $\mathbb{R}^N$ $\quad (x,y) = \sum_{i=1}^{N} x_i y_i$ Hilbert

$x = (x_1, x_2 \dots x_N) \qquad y = (y_1, y_2, \dots y_N)$

4) $L^2(\Omega)$ $\quad (f,g) = \int_\Omega \underbrace{f(x) g(x)}_{L^1(\Omega) \text{ by Hölder's inequality}} dx$ Hilbert

5) later Sobolev spaces $W^{1,2}(\Omega)$, $W_0^{1,2}(\Omega)$

Hilbert

6) $C([a,b])$ $\quad (f,g) = \int_\Omega f(x) g(x)\, dx$ Not Hilbert

DEF

A space with scalar product is called a Hilbert space if it is complete w.r. to the metric induced by the norm.

Let H be a vector space with a scalar product

We define $\quad \|x\| = \sqrt{(x,x)}$

Let us check that it is a norm.

1) $\|x\| \geq 0$

$\|x\| = 0 \iff x = 0$

ok by def. of scalar product

2) $\|\lambda x\| = |\lambda| \, \|x\|$ $\qquad x \in H, \ \lambda \in \mathbb{R}$

$\|\lambda x\| = \sqrt{(\lambda x, \lambda x)} = \sqrt{\lambda^2 (x,x)} =$

$$= |\lambda| \sqrt{(x,x)} = |\lambda| \, \|x\|$$

3) Triangle inequality

$$\|x+y\| \leq \|x\| + \|y\|$$

We need the $\boxed{\text{Cauchy - Schwarz inequality} \quad |(x,y)| \leq \|x\| \, \|y\|}$

Proof $0 \leq \|x+ty\|^2 = (x+ty, x+ty) =$ $\qquad t \in \mathbb{R}$

$$= \|x\|^2 + 2t(x,y) + t^2\|y\|^2 \qquad \forall t \in \mathbb{R}.$$

$$\Rightarrow \underset{4(x,y)^2 - 4\|x\|^2\|y\|^2}{\underset{\parallel}{\Delta}} \leq 0$$

$$\Rightarrow |x,y| \leq \|x\| \, \|y\|. \qquad \square$$

Proof of the triangle inequality of the norm.

$$\|x+y\| \overset{?}{\leq} \|x\| + \|y\|$$

$$\Updownarrow$$

$$\overset{(x+y,\, x+y)}{\overset{\parallel}{\|x+y\|^2}} \overset{?}{\leq} (\|x\| + \|y\|)^2$$

$$\cancel{\|x\|^2} + 2(x,y) + \cancel{\|y\|^2} \overset{?}{\leq} \cancel{\|x\|^2} + \cancel{\|y\|^2} + 2\|x\| \, \|y\|$$

$$(x,y) \leq \|x\| \, \|y\|$$

this is true by Schwarz's inequality

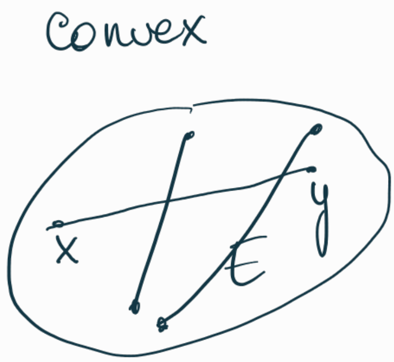
Convex sets

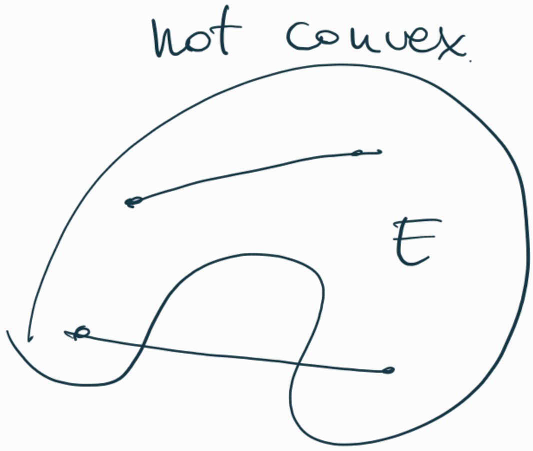
$E \subseteq V$ vector space

E is said to be <u>convex</u> if $\forall x, y \in E$ $\forall t \in [0,1]$

$$tx + (1-t)y \in E$$

this describes the segment which goes from y to x.





Rem If $M \subset V$ is a vector subspace, it is also convex

We recall that in a metric space X a set E is <u>closed</u> if one of the two equivalent statements hold

i) if $\{x_n\} \subset E$, $x_n \to x \Rightarrow x \in E$

ii) $X \setminus E$ is open, i.e.
$\forall x \in X \setminus E$ there is a neighborhood of x all contained in $X \setminus E$.

THEOREM let H be an Hilbert space. let $E \subset H$ be a closed and convex set. Then E admits a unique point x of minimum norm.