THEOREM (Banach theorem)

Let (X,d) be a complete metric space

Let $F: X \to X$ be a contraction.

Then $\exists!\ \bar{x} \in X$ s.t. $F(\bar{x}) = \bar{x}$

that is, $\bar{x}$ is a fixed point of F

that is:

$$d(F(x),F(y)) \leq \alpha\, d(x,y) \quad \forall x,y \in X, \quad \boxed{\alpha < 1}$$

Proof Let x_0 be any point of X.

I define a sequence

$$x_1 = F(x_0), \quad x_2 = F(x_1), \quad \dots \quad x_{n+1} = F(x_n)$$

I will show that.

(i) $\{x_n\}$ is a Cauchy sequence

(ii) by completeness of X, $x_n \longrightarrow \bar{x} \in X$

(iii) $\bar{x}$ is the fixed point of F $\quad F(\bar{x}) = \bar{x}$

(iv) The fixed point is unique

(i) I want to prove that $\forall \varepsilon > 0\ \exists \bar{n}$ s.t.

$$\forall n,m \geq \bar{n} \qquad d(x_n, x_m) < \varepsilon.$$

If we assume $m \geq n$, I can write $m = n+p \quad p \in \mathbb{N}$.

so I want to prove that $\forall \varepsilon > 0\ \exists \bar{n}$ s.t.

$$\forall n \geq \bar{n},\ \forall p \in \mathbb{N} \qquad d(x_{n+p}, x_n) < \varepsilon$$

$$d(\underset{F(x_1)}{\underset{\shortparallel}{x_2}}, \underset{F(x_0)}{\underset{\shortparallel}{x_1}}) = d(F(x_1), F(x_0)) \le \alpha\, d(x_1, x_0).$$

$$d(x_3, x_2) \le \alpha\, d(x_2, x_1) \le \alpha^2 d(x_1, x_0)$$

$$d(x_{n+1}, x_n) \le \alpha^n d(x_1, x_0)$$

$$d(x_{n+p}, x_n) \le \underset{\alpha^n d(x_1,x_0)}{\underset{\wedge\!\shortparallel}{d(x_n, x_{n+1})}} + \underset{\alpha^{n+1} d(x_1,x_0)}{\underset{\wedge\!\shortparallel}{d(x_{n+1}, x_{n+2})}} + \dots + \underset{\alpha^{n+p-1} d(x_1,x_0)}{\underset{\wedge\!\shortparallel}{d(x_{n+p-1}, x_{n+p})}}$$

$$\le d(x_1, x_0) \sum_{k=n}^{n+p-1} \alpha^k < d(x_1, x_0) \boxed{\sum_{k=n}^{\infty} \alpha^k}$$

small if n is large enough. because it is the error of a converging sequence

ii) by completeness, every Cauchy sequence converges

so $x_n \to \bar{x} \in X$.

iii) $$\underset{\bar{x}}{\underset{\downarrow}{x_{n+1}}} = \underset{F(\bar{x})}{\underset{\downarrow}{F(x_n)}}$$

$n \to +\infty$ by continuity (which is implied by the def of contraction

Uniqueness of limits $\Rightarrow$ $\bar{x} = F(\bar{x})$

true in all metric spaces.

iv) Assume that $\bar{x}, \bar{y} \in X$ are fixed points of F. $F(\bar{x}) = \bar{x}$, $F(\bar{y}) = \bar{y}$.

$$d\big(\underbrace{F(\bar{x})}_{=\bar{x}}, \underbrace{F(\bar{y})}_{=\bar{y}}\big) \leq \alpha\, d(\bar{x}, \bar{y}).$$

$$d(\bar{x}, \bar{y}) \leq \alpha\, d(\bar{x}, \bar{y}).$$

$$d(\bar{x}, \bar{y})\underbrace{(1-\alpha)}_{>0} \leq 0$$

$$\Downarrow$$

$$d(\bar{x}, \bar{y}) = 0 \Rightarrow \bar{x} = \bar{y} \qquad \square$$

__Remark__ The proof shows how to find the fixed point.

1) $x_0 \in X \Rightarrow x_{n+1} = F(x_n)$

2) this sequence converges __rapidly__ to the fixed point.

$\square$

Theorem

$$(P) \qquad \begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$$

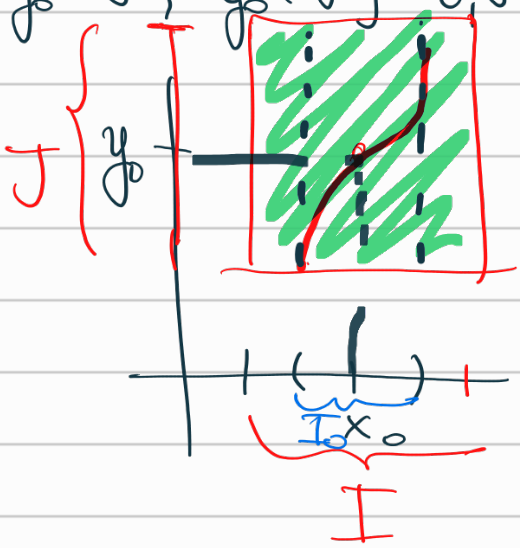
$f : I \times J \longrightarrow \mathbb{R}$ of class $C^1(I \times J)$

$I = [x_0 - \delta, x_0 + \delta]$, $J = [y_0 - \sigma, y_0 + \sigma]$ $\delta, \sigma > 0$

Then $\exists$ $I_0 = [x_0 - r, x_0 + r]$ and

$\exists!$ $y(x) : I_0 \longrightarrow J$ of class C^1

which is solution of (P)



Proof

1) The original problem

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases} \quad y \in C^1(I_0; J) \qquad (P)$$

is equivalent to the integral problem

(Q) find $y(x) \in C^0(I_0; J)$ such that

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \quad \forall x \in I_0$$

2) To solve problem (Q), we consider the metric space

$$X = C^0(I_0; J) = \{ y(x) \in C^0(I_0) : |y(x) - y_0| \le \sigma \ \forall x \in I_0 \}$$

endowed with the distance. $I_0 = [x_0 - r, x_0 + r]$

$$d(y, z) = \|y - z\|_\infty = \max_{x \in I_0} |y(x) - z(x)|$$

This is a complete metric space. (Easy to prove)

We consider the following operator

$$F : X \longrightarrow X$$
$$y \longmapsto F(y) =: w$$

where $w(x) = (F(y))(x)$ is defined by

$$(F(y))(x) = w(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

So, solving (Q) is equivalent to show that

$\exists!\ y \in X$ which is a fixed point of F,

that is, $F(y) = y$.

By choosing $I_0 = [x_0 - r, x_0 + r]$,

that is, by choosing $r > 0$,

we want to check that

i) $F : X \longrightarrow X$.

It is clear that $F(y) \in C^0(I_0)$, but we have to check that

$$|(F(y))(x) - y_0| \le \sigma \quad \forall x \in I_0$$

Last time we checked that this is true if we choose

$$\boxed{r \leq \frac{\sigma}{M}}, \text{ where } M = \max_{I \times J} |f(x,y)|$$

ii) Finally, we want to check that

$F : X \to X$ is a contraction, i.e.

$$d(F(y), F(z)) \leq \alpha \, d(y,z) \quad \forall y, z \in X.$$

Then we can use Banach's theorem and conclude that $\exists !$ $y \in X$, fixed point of F.

$$d(F(y), F(z)) = \max_{x \in I_0} |(F(y))(x) - (F(z))(x)|$$

$$|F(y)(x) - F(z)(x)| = \left| \cancel{y_0} + \int_{x_0}^{x} f(t, y(t))\,dt - \cancel{y_0} - \int_{x_0}^{x} f(t, z(t))\,dt \right| =$$

$$= \left| \int_{x_0}^{x} [f(t, y(t)) - f(t, z(t))]\,dt \right| \leq (\text{triangle ineq.})$$

$$\leq \left| \int_{x_0}^{x} |f(t, y(t)) - f(t, z(t))|\,dt \right| \leq$$

Remark

$$|f(t, y(t)) - f(t, z(t))| = \underbrace{\left| \frac{\partial f}{\partial y}(t, \xi) \right|}_{\leq L \text{ by Weierstrass' thm.}} \underbrace{|y(t) - z(t)|}_{\leq d(y,z)} \leq$$

for some $\xi \in I_0$

$$\leq L \, d(y,z).$$

$$\leq L\, d(y,z)\, |x-x_0| \leq Lr\, d(y,z)$$

Taking the max in x

$$d\big(F(y), F(z)\big) \leq \underbrace{Lr}_{\alpha}\, d(y,z)$$

$\alpha < 1$ if we choose $\boxed{r < \frac{1}{L}}$

In the end, everything works if we choose

$$r < \min\left\{\frac{\delta}{M}, \frac{1}{L}\right\}$$

So F is a contraction. End of proof.

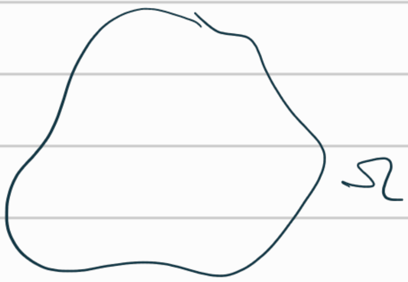
Later,

We will apply this same theorem for some problems of the form

known

$$\begin{cases} \Delta u = g(x) & \text{in } \Omega \text{ open set} \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\Delta u(x,y) = \nabla^2 u(x,y) = u_{xx}(x,y) + u_{yy}(x,y)$$



$$\begin{cases} \Delta u = g(x,u) \\ u = 0 \end{cases}$$

L^p spaces

References: Royden: Real Analysis
Brezis: Functional Analysis

Let $\Omega \subseteq \mathbb{R}^N$ measurable. $N = 1, 2, \dots$

$1 \leq p \leq \infty$. $L^p(\Omega)$

The most important cases are $p=1$, $p=2$, $p=+\infty$

Def Let $1 \leq p < \infty$
We call $\mathcal{L}^p(\Omega) = \{ f : \Omega \to \overline{\mathbb{R}}$ Lebesgue-measurable. s.t. $\int_\Omega |f(x)|^p dx < \infty \}$

Remark $\mathcal{L}^p(\Omega)$ is a vector space.

$f \in \mathcal{L}^p(\Omega)$, $c \in \mathbb{R}$ $\quad \overset{?}{\Longrightarrow} \quad cf \in \mathcal{L}^p(\Omega)$

$$\int_\Omega |c f(x)|^p dx = |c|^p \int_\Omega |f(x)|^p dx < \infty$$

$$f, g \in \mathcal{L}^p(\Omega) \overset{?}{\Longrightarrow} f+g \in \mathcal{L}^p(\Omega)$$

$$\int_\Omega |f(x)+g(x)|^p dx \overset{?}{<} \infty$$

$$\wedge\!\!|$$

$$\int_\Omega (|f(x)|+|g(x)|)^p dx \leq 2^p \int_\Omega (|f(x)|^p + |g(x)|^p) dx < \infty$$

I want to show that $(a+b)^p \le 2^p (a^p + b^p)$

$\forall a, b \ge 0$

$$
\begin{aligned}
a &= (a^p)^{1/p} \le (a^p + b^p)^{1/p} \\
b &\le (a^p + b^p)^{1/p} \\
\hline
a+b &\le 2(a^p+b^p)^{1/p} \\
(a+b)^p &\le 2^p (a^p + b^p)
\end{aligned}
$$

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on this vector space we introduce a norm.

$$\|f\|_p = \left[\int_\Omega |f(x)|^p \, dx\right]^{1/p}$$

Is it a norm?

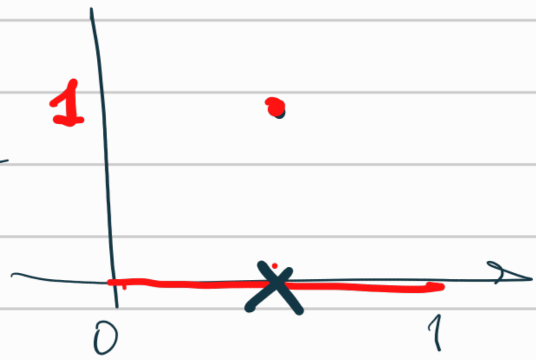
1) $\|f\|_p = 0 \iff f \equiv 0$ false!

2) $\|c f\|_p = |c| \, \|f\|_p$ trivial $\forall f \in \mathcal{L}^p(\Omega)$ $\forall c \in \mathbb{R}$

3) $\|f+g\|_p \le \|f\|_p + \|g\|_p$ true, but not trivial

Example $\Omega = (0,1)$

$$f(x) = \begin{cases} 1 & \text{if } x = 1/2 \\ 0 & \text{otherwise} \end{cases}$$



This f has L^p norm zero,

but it is not identically zero.

Also if $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

$$\|f\|_{L^p} = 0$$

So if $\|f\|_p = 0 \not\Rightarrow f(x) = 0 \quad \forall x \in \Omega$.

But we can prove that

$$\|f\|_p = 0 \Longrightarrow f(x) = 0 \quad \text{a.e. } x \in \Omega.$$

i.e. the set where $f(x) \neq 0$ has zero measure.

Assume

$$\int_\Omega |f(x)|^p \, dx = 0$$

$$\{x \in \Omega : f(x) \neq 0\} = \{x \in \Omega : |f(x)| > 0\} =$$

$$= \bigcup_{n=1}^{\infty} \left\{x \in \Omega : |f(x)| > \frac{1}{n}\right\}$$

countable subadditity.

$$m\{f(x) \neq 0\} \leq \sum_{n=1}^{\infty} \underbrace{m\left\{|f(x)| > \frac{1}{n}\right\}}_{= 0} = 0$$

Moreover.

$$0 = \int_\Omega |f(x)|^p dx \geq \int_{\{|f(x)| > \frac{1}{n}\}} |f(x)|^p dx \geq \int_{\{\ \}} \frac{1}{n^p} dx = \frac{1}{n^p} m\left\{|f(x)| > \frac{1}{n}\right\}$$

$$\Rightarrow m\left\{x \in \Omega : |f(x)| > \frac{1}{n}\right\} = 0$$

⇒ To overcome this problem, we identify two functions if they are the same except on a set of zero measure.

If $f(x) = \tilde{f}(x)$ a.e. on Ω, we consider it to be the same function.

We now define

$L^p(\Omega) = \mathcal{L}^p(\Omega)$ after identifying functions which are equal a.e.

$$= \mathcal{L}^p(\Omega) \Big/ f(x) = \tilde{f}(x) \text{ a.e.}$$

Now the norm

$$\|f\|_p = \left[\int_\Omega |f(x)|^p dx\right]^{1/p} \quad \text{is a norm.}$$

So $\left(L^p(\Omega), \|f\|_p\right)$ is a normed space: the L^p space $L^p(\Omega)$

<u>Remark</u> if $f \in L^p(\Omega)$, $x_0 \in \Omega$

then $f(x_0)$ has no meaning

$\mathcal{L}^\infty(\Omega) = \{ f(x) : \Omega \to \overline{\mathbb{R}}$ measurable s.t.

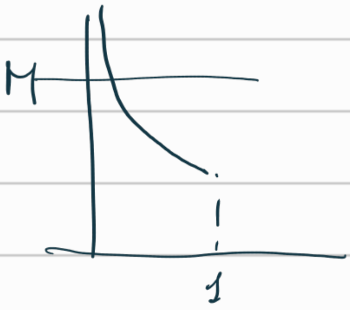
$\exists M \geq 0$ s.t. $|f(x)| \leq M$ for a.e. $x \in \Omega \}$

i.e. the set $\{x \in \Omega : |f(x)| > M\}$ has measure zero.

we will say "f is essentially bounded".

Example.

$\Omega = (0,1) \qquad f(x) = \frac{1}{x}$



$f \overset{?}{\in} \mathcal{L}^{\infty}(0,1)$ No!

If $f \in \mathcal{L}^{\infty}(\Omega)$, we define

$$\|f\|_{\infty} = \inf \{ M \geq 0 : |f(x)| \leq M \text{ a.e. in } \Omega \}$$

(simple exercise) $= \underline{\underline{\min}} \{ M \geq 0 : |f(x)| \leq M \text{ a.e. in } \Omega \}$

$\Rightarrow \quad |f(x)| \leq \|f\|_{\infty}$ for a.e. $x \in \Omega$.

As before, we have to identify functions which are equal a.e on Ω

$L^{\infty}(\Omega) = \mathcal{L}^{\infty}(\Omega)$ after this identification is made

$\|f\|_{\infty}$ is a norm.

(i) $\|f\|_{\infty} = 0 \Rightarrow f \equiv 0$ (a.e.)

(ii) $\|\lambda f\|_{\infty} = |\lambda| \, \|f\|_{\infty}$

(iii) $\|f+g\|_{\infty} \leq \|f\|_{\infty} + \|g\|_{\infty}$.

Remark Even if $f(x_0)$ has no meaning for fixed x_0, the sum of $f+g$ has a meaning.

If $\begin{array}{l} f = \tilde{f} \text{ a.e.} \\ g = \tilde{g} \text{ a.e} \end{array} \Bigg| \longrightarrow f+g = \tilde{f}+\tilde{g}$ a.e.

Let $p \in [1, +\infty]$. We define its conjugate Hölder's exponent as that number $q \in [1, +\infty]$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

For instance

$$\text{If } p = 1 \implies q = \infty$$

$$1 < p < \infty \implies \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \implies \boxed{q = \frac{p}{p-1}}$$

$$\text{If } p = \infty \implies q = 1$$

$$\text{If } p = 2 \implies q = 2$$

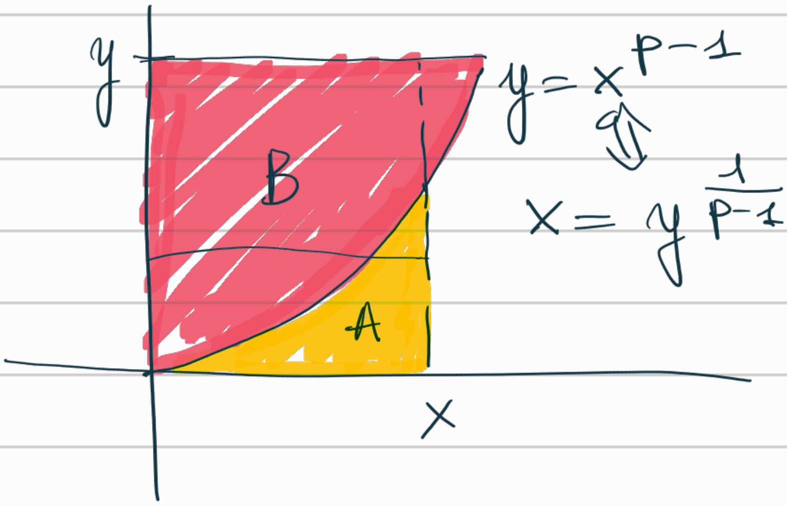
$$p = 3 \implies q = \frac{3}{2}$$

<u>Young's inequality</u> (in $\mathbb{R}$)

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q} \qquad \begin{array}{l} \forall x, y \geq 0 \\ \forall p, q \in (1, +\infty) \\ \text{s.t. } \frac{1}{p} + \frac{1}{q} = 1 \end{array}$$

If $p = q = 2$ it is trivial $\quad xy \leq \frac{x^2}{2} + \frac{y^2}{2}$

Proof. We take



$$\text{area } A = \int_0^x t^{p-1}\,dt = \frac{x^p}{p}$$

$$\text{area } B = \int_0^y t^{\frac{1}{p-1}}\,dt = \frac{y^{\frac{1}{p-1}+1}}{\frac{1}{p-1}+1} = \frac{y^q}{q}$$

$$\frac{1}{p-1} + 1 = \frac{\cancel{1} + p - \cancel{1}}{p-1} = q$$

From the drawing we obtain

$$\underset{\substack{\shortparallel \\ \text{area of the rectangle}}}{xy} \le \text{area } A + \text{area } B = \frac{x^p}{p} + \frac{y^q}{q}.$$

with equality if and only if

$$y = x^{p-1}$$

Hölder's inequality

Let $p, q \in [1, +\infty]$ conjugate $\left(\frac{1}{p} + \frac{1}{q} = 1 \right)$

Let $f \in L^p(\Omega)$, $g \in L^q(\Omega)$. Then the product $f \cdot g$ belongs to $L^1(\Omega)$

$$\underbrace{\int_\Omega |fg| \, dx}_{= \|fg\|_1} \leq \|f\|_{L^p} \, \|g\|_{L^q}$$

Special case $p = q = 2$

$f, g \in L^2(\Omega) \Rightarrow fg \in L^1(\Omega)$

$$\int_\Omega |fg| \, dx \leq \left[\int |f|^2 dx \right]^{1/2} \left[\int |g|^2 dx \right]^{1/2}$$

Proof.

1° case $p = \infty$, $q = 1$ or viceversa. (easy)

$$\left. \begin{array}{l} f \in L^\infty(\Omega) \\ g \in L^1(\Omega) \end{array} \right| \overset{?}{\Longrightarrow} \int_\Omega |fg| \, dx \leq \|f\|_{L^\infty} \|g\|_{L^1}.$$

$$\int_\Omega \underbrace{|f(x)||g(x)|}_{\leq \, \|f\|_\infty |g(x)| \text{ a.e.}} dx \leq \int_\Omega \|f\|_\infty |g(x)| \, dx = \|f\|_\infty \int_\Omega |g(x)| dx = \|f\|_\infty \|g\|_1$$

$|f(x)| \leq \|f\|_\infty$ a.e. in Ω.

2° case (more difficult) $\qquad 1 < p, q < \infty$

$$q = \frac{p}{p-1}$$

Young's ineq.

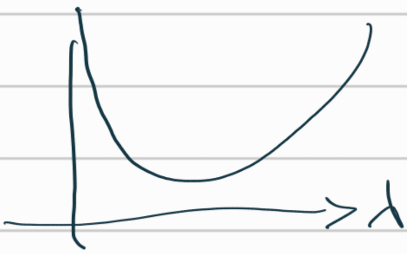
$$\int_\Omega |f(x)||g(x)|\,dx \leq \frac{1}{p}\int_\Omega |f(x)|^p\,dx + \frac{1}{q}\int_\Omega |g(x)|^q\,dx$$

I take $\lambda f(x)$ instead of $f(x)$

$$\lambda \int_\Omega |f(x)||g(x)|\,dx \leq \frac{\lambda^p}{p}\int_\Omega |f(x)|^p\,dx + \frac{1}{q}\int_\Omega |g(x)|^q\,dx$$

divide by $\lambda > 0$

$$\int_\Omega |f(x)||g(x)|\,dx \leq \underbrace{\frac{\lambda^{p-1}}{p}\int_\Omega |f(x)|^p\,dx + \frac{1}{q\lambda}\int_\Omega |g(x)|^q\,dx}_{\varphi(\lambda)}$$



$$\varphi(\lambda) = c_1 \lambda^{p-1} + \frac{c_2}{\lambda}$$

where $c_1 = \dfrac{\|f\|_p^p}{p} \qquad c_2 = \dfrac{\|g\|_q^q}{q}$

I want to minimize $\varphi(\lambda)$.

$$\varphi'(\lambda) = c_1(p-1)\lambda^{p-2} - \frac{c_2}{\lambda^2} = 0$$

$$\Rightarrow \quad \lambda^p = \frac{c_2}{c_1(p-1)} = \frac{\|g\|_q^q\,\cancel{p}}{\cancel{q}\,\|f\|_p^p\,\cancel{(p-1)}}$$

$$\Rightarrow \lambda = \frac{\|g\|_q^{q/p}}{\|f\|_p} \qquad \frac{p}{(p-1)p} = \frac{1}{p-1}$$

I now plug $\boxed{\lambda = \frac{\|g\|_q^{\frac{1}{p-1}}}{\|f\|_p}}$ in the expression of $\varphi(\lambda)$

$$\int_\Omega |f(x)||g(x)|\,dx \leq \underbrace{\frac{\lambda^{p-1}}{p}\int_\Omega |f(x)|^p dx + \frac{1}{q\lambda}\int_\Omega |g(x)|^q dx}_{\varphi(\lambda)}$$

$$\int_\Omega |f(x)g(x)|\,dx \leq \underbrace{\frac{1}{p}\frac{\|g\|_q}{\cancel{\|f\|_p^{p-1}}}\|f\|_p^{\cancel{p}}}_{\frac{1}{p}\|f\|_p\|g\|_q} + \frac{1}{q}\|f\|_p\|g\|_q =$$

$$= \underbrace{\left(\frac{1}{p}+\frac{1}{q}\right)}_{1}\|f\|_p\,\|g\|_q)$$