In $\mathbb{R}^N$ it can be shown that all norms are "equivalent", i.e., If $\|\cdot\|$ $|||\cdot|||$ are norms on $\mathbb{R}^N$, then there exist $c_1, c_2 > 0$ such that

$$c_1 \|x\| \le |||x||| \le c_2 \|x\| \qquad \forall x \in X$$

This means that inside a neighboorhood of x_0 in one norm we can find a neighborhood of x_0 in the other norm. So, for instance, the definition of converging sequences does not depende on the choice of norm.

In practice, it is easy to see that

$$(x_n, y_n) \longrightarrow (x,y) \text{ in } \mathbb{R}^2 \iff \begin{cases} x_n \to x & \text{in } \mathbb{R} \\ y_n \to y & \text{in } \mathbb{R} \end{cases}$$

But in general, when you consider <u>general normed spaces,</u> the situation is very different if you choose one norm or another (see Examples later).

<u>Remark</u> All normed spaces are metric spaces.

There are many metric spaces which are not normed spaces

<u>Examples</u>

1) $E \subseteq \mathbb{R}$, for instance $E = [a,b]$. with the usual distance $d(x,y) = |x-y|$

2) $\mathbb{R}$ $$d(x,y) = \frac{|x-y|}{1 + |x-y|}$$

Example of normed vector spaces <u>of functions</u>.

1) $X = C^0([a,b]) = \{ f : [a,b] \to \mathbb{R} \text{ continuous} \}$

it is a vector space.

If $f \in C^0([a,b])$, we define

Weierstrass' theorem

$$\|f\| = \|f\|_\infty = \sup_{x\in[a,b]} |f(x)| = \max_{x\in[a,b]} |f(x)| \in [0,+\infty)$$

with this norm $C^0([a,b])$ is complete

1) $\|f\|_\infty = 0 \iff f \equiv 0$ in $[a,b]$.

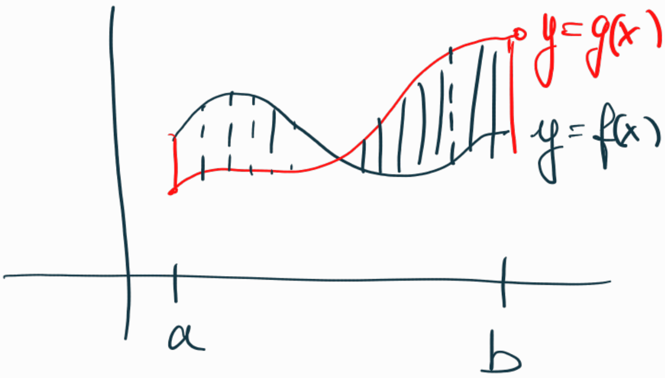
2) $\|\lambda f\|_\infty = \max |\lambda| |f(x)| = |\lambda| \max |f(x)| = |\lambda| \, \|f\|_\infty$

3) $|f(x)+g(x)| \le \underset{\|f\|_\infty}{\underset{\wedge|}{|f(x)|}} + \underset{\|g\|_\infty}{\underset{\wedge|}{|g(x)|}} \le \|f\|_\infty + \|g\|_\infty$

$\forall x \in [a,b]$

$$\Rightarrow \quad \|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty .$$

$$\|f-g\|_\infty = \max_{x\in[a,b]} |f(x)-g(x)|$$

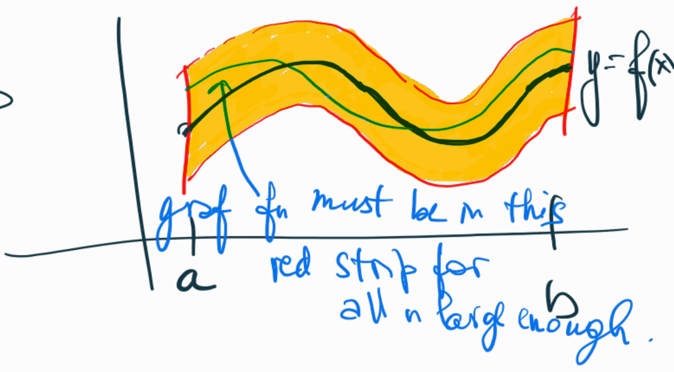


So, $f_n \to f$ in $C^0([a,b])$ means.

$$\max |f_n(x) - f(x)| \xrightarrow{n\to+\infty} 0$$

This is also called

<u>"uniform" convergence</u>



Remark If $f_n \to f$ in $C^0([a,b])$ in this norm,

then
$$\max_{x\in[a,b]} |f_n(x)-f(x)| \xrightarrow{n\to+\infty} 0$$

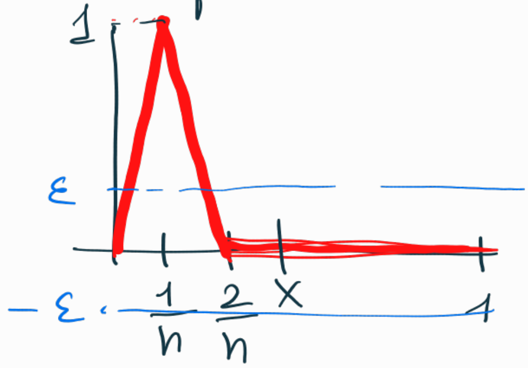
This implies that $|f_n(x)-f(x)| \xrightarrow{n\to+\infty} 0 \quad \forall x\in[a,b]$

i.e. "$f_n(x) \to f(x) \quad \forall x$"

Therefore: uniform convergence implies pointwise convergence

The opposite implication is false.

Take $f_n(x) =$



Then
$$f_n(x) \longrightarrow 0 \qquad \forall x\in[0,1]$$

but
$$\|f_n - 0\|_\infty = \max_{x\in[0,1]} |f_n(x)| = 1 \not\to 0$$

This is not the only norm we could use on $X = C^0([a,b])$

For instance we could use the norm

$$\|f\|_1 = \int_a^b |f(x)|\,dx$$

$$\|f\|_2 = \left[\int_a^b |f(x)|^2\,dx\right]^{\frac{1}{2}}$$

we will check later that these are norms

With these norms $C^0([a,b])$ is not complete

Another example of normed space

$$X = C^1([a,b]) = \{ f:[a,b]\to \mathbb{R},\ f \text{ is differentiable in } [a,b],\ f'(x) \text{ continuous} \}$$

$$\subset C^0([a,b])$$

it is a vector space.

We could use one of the following norms:

1) $\|f\|_\infty = \max\limits_{x\in[a,b]} |f(x)|$ (same as before) (not complete)

2) $\|f\|_{1,\infty} = \|f\|_\infty + \|f'\|_\infty = \max\limits_{x\in[a,b]} |f(x)| + \max\limits_{''} |f'(x)|$ (complete)

3) $\|f\|_1 = \int_a^b |f(x)|\,dx$ (not complete)

4) $\|f\|_2 = \left[\int_a^b |f(x)|^2\,dx\right]^{1/2}$ (not complete)

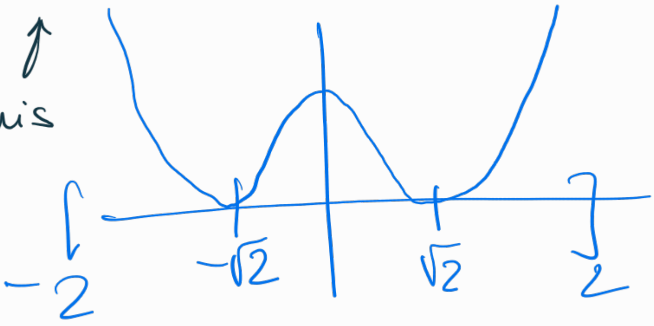
A big difference between norms: COMPLETENESS.

For instance, $\mathbb{R}$ is complete, $\mathbb{Q}$ is not complete (see def. below)

This implies that Mathematical Analysis in $\mathbb{Q}$ is unsatisfactory:

For instance, if we work in $\mathbb{Q}$, Weierstrass' theorem does not hold. For instance, this polynomial does not has a minimum in $[-2,2]$ ($\sqrt{2}$ is irrational)

$f(x) = (x^2-2)^2$ in $[-2,2]$.



Complete metric spaces.

Let (X,d) a metric space.

A sequence $\{x_n\} \subset X$ is called a Cauchy sequence if $\forall \varepsilon > 0 \ \exists \bar{n}$ s.t. $\forall m,n \geq \bar{n} \ d(x_n, x_m) < \varepsilon$

(That is, elements of the sequence are "close to each other" when the indexes are large.

It is easy to see that every converging sequence is also a Cauchy sequence

If $x_n \to x$ in X

$$\|x_n - x_m\| \leq \underbrace{\|x_n - x\|}_{\varepsilon/2} + \underbrace{\|x - x_m\|}_{\varepsilon/2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

if n, m are large

DEF

A metric space (X,d) is called complete if every Cauchy sequence is also convergent.

i.e. for every $\{x_n\}$ Cauchy $\exists \ \bar{x} \in X$ s.t $x_n \to \bar{x}$

For instance, $(\mathbb{R}, |x-y|)$ is complete

$(\mathbb{Q}, |x-y|)$ is not complete

For instance, we take a sequence of rational numbers which converges to $\sqrt{2}$

It is a Cauchy sequence, but it does not converge to any point in $\mathbb{Q}$.

Another example

$$X = (a,b) \qquad d(x,y) = |x-y|$$

This is not a complete metric space. Take the sequence

$x_n = a + \frac{1}{n}$ is Cauchy sequence.

but it has no limit belonging to $X = (a,b)$.

Theorem $C^0([a,b])$ with the norm

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$ is a complete normed space

Dim. Let $\{f_n\}$ be a Cauchy sequence. We must prove that $\exists\, f \in C^0([a,b])$ s.t. $\|f_n - f\|_\infty \xrightarrow{n\to\infty} 0$

$\{f_n\}$ is Cauchy in $C^0([a,b]) \Longleftrightarrow$

$$\forall \varepsilon > 0 \quad \exists\, n_\varepsilon \text{ s.t. } \quad \underbrace{\|f_n - f_m\|_\infty}_{\displaystyle \| \atop \displaystyle \max_{x\in[a,b]} |f_n(x) - f_m(x)| \atop \displaystyle \vee\!\!/ \atop \displaystyle |f_n(x)-f_m(x)| \;\; \forall x \in [a,b]} < \varepsilon \quad \forall n,m \geq n_\varepsilon$$

$\Rightarrow \forall x \in [a,b] \quad \{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$.

$\Rightarrow \quad f_n(x) \to f(x)$ because $\mathbb{R}$ is complete.

Therefore $f_n \to f$ pointwise.

We must show that

1) $\|f_n - f\|_\infty \xrightarrow{n\to+\infty} 0$

2) $f \in C^0([a,b])$

Proof of (1)

We know that

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \text{ s.t. } \forall n,m > n_\varepsilon \quad \max_{x\in[a,b]} |f_n(x) - f_m(x)| \leq \varepsilon$$

$$\Updownarrow$$

$$|f_n(x) - f_m(x)| \leq \varepsilon \quad \forall x \in [a,b]$$

Now, let us take $m \to +\infty \Rightarrow f_m(x) \to f(x)$

$$\Rightarrow \quad |f_n(x) - f(x)| \leq \varepsilon \quad \forall x\in[a,b] \quad \forall n \geq n_\varepsilon$$

$$\Rightarrow \quad \|f_n - f\|_\infty \leq \varepsilon \quad \forall n \geq n_\varepsilon$$

2) f is continuous.

$$|f(x) - f(x_0)| = |(f(x) - f_n(x)) + (f_n(x) - f_n(x_0)) + (f_n(x_0) - f(x_0))|$$

$$\leq \underbrace{\underbrace{|f(x) - f_n(x)|}_{\leq \|f - f_n\|_\infty}}_{< \varepsilon/3} + |f_n(x) - f_n(x_0)| + \underbrace{\underbrace{|f_n(x_0) - f(x_0)|}_{\leq \|f-f_n\|_\infty}}_{< \varepsilon/3}$$

if n is large enough

We fix n s.t. the previous (red) inequalities are true

$$|f(x)-f(x_0)| < \frac{\varepsilon}{3} + \underbrace{|f_n(x)-f_n(x_0)|}_{\wedge \ \varepsilon/3 \text{ if } |x-x_0| \text{ is small enough}} + \frac{\varepsilon}{3} < \varepsilon$$

(n is fixed)

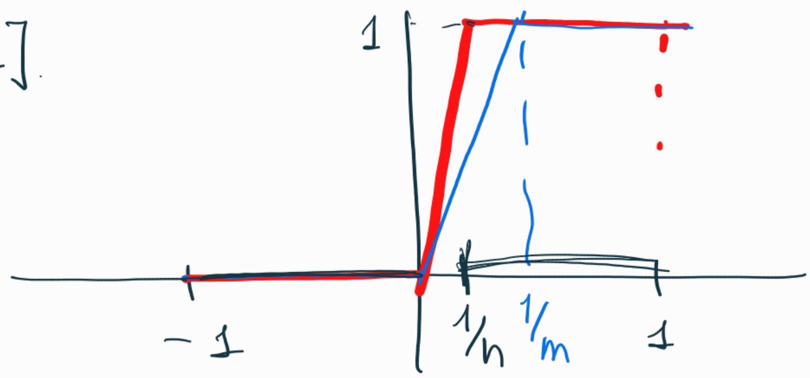
$\square$

Remark $C^0([a,b])$ with the L^1-norm:

$\|f\|_1 = \int_a^b |f(x)|\,dx$ is not complete.

Take $[a,b]=[-1,1]$.

$f_n(x) =$



It is a Cauchy sequence:

$$\|f_n - f_m\|_1 = \int_{-1}^{1} |f_n(x)-f_m(x)|\,dx = \int_0^{1/m} \underbrace{|f_n(x)-f_m(x)|}_{\leq 1}\,dx \leq \qquad n \geq m$$

$$\leq \frac{1}{m} < \varepsilon \qquad \text{if } n \geq m > \varepsilon$$

But f_n cannot converge to a continuous function

Assume that $\|f_n - f\|_1 \xrightarrow{n\to+\infty} 0$, with f continuous.

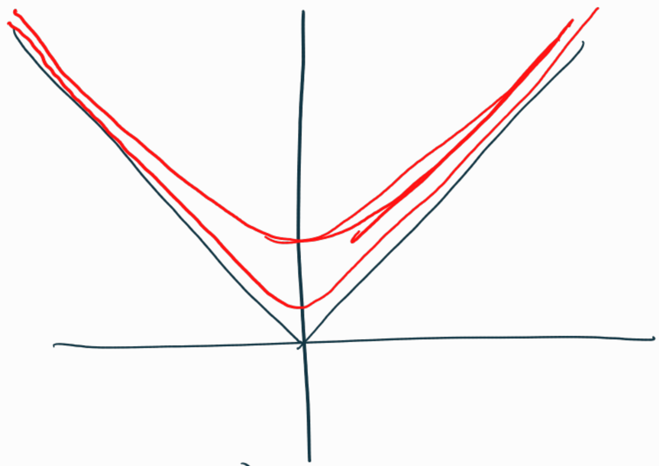
It follows that $f(x)=0 \quad \forall x \in [-1,0]$

$f(x)=1 \quad \forall x \in [\varepsilon,1] \quad \forall \varepsilon>0$

So f has a jump in $x=0 \Rightarrow$ it is not continuous. $\lightning$

Remark $C^1([a,b])$ with the norm

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$ is not complete.



$[a,b]=[-1,1]$

$$f_n(x) = \sqrt{x^2+\frac{1}{n}}$$

$f_n(x) \to |x|$ pointwise, and uniformly,

but $f(x)=|x|$ is not in $C^1([-1,1])$

Exercise: Prove that $C^1([a,b])$ is complete with the norm $\|f\|_{1,\infty} = \|f\|_\infty + \|f'\|_\infty$

Definition Let (X,d) be a metric space. A function $F: X \to X$ is called a **contraction** if $\exists\, \alpha \in (0,1)$ s.t.

$$d(F(x),F(y)) \le \alpha\, d(x,y) \quad \forall x,y\in X.$$

For instance $(X,d) = (\mathbb{R}, |x-y|)$

$$F(x) = 5+\sin\left(\frac{x}{2}\right)$$

$$|F(x)-F(y)| = \left|\cancel{5}+\sin\left(\frac{x}{2}\right)-\cancel{5}-\sin\left(\frac{y}{2}\right)\right| =$$

$$= \underbrace{\left|\frac{1}{2}\cos\left(\frac{\xi}{2}\right)\right|}_{F'(\xi)} |x-y| \leq \frac{1}{2}|x-y|$$

THEOREM (Banach theorem)

Let (X,d) be a <u>complete</u> metric space

Let $F : X \to X$ be a contraction.

Then $\exists!\ \bar{x} \in X$ s.t. $F(\bar{x}) = \bar{x}$

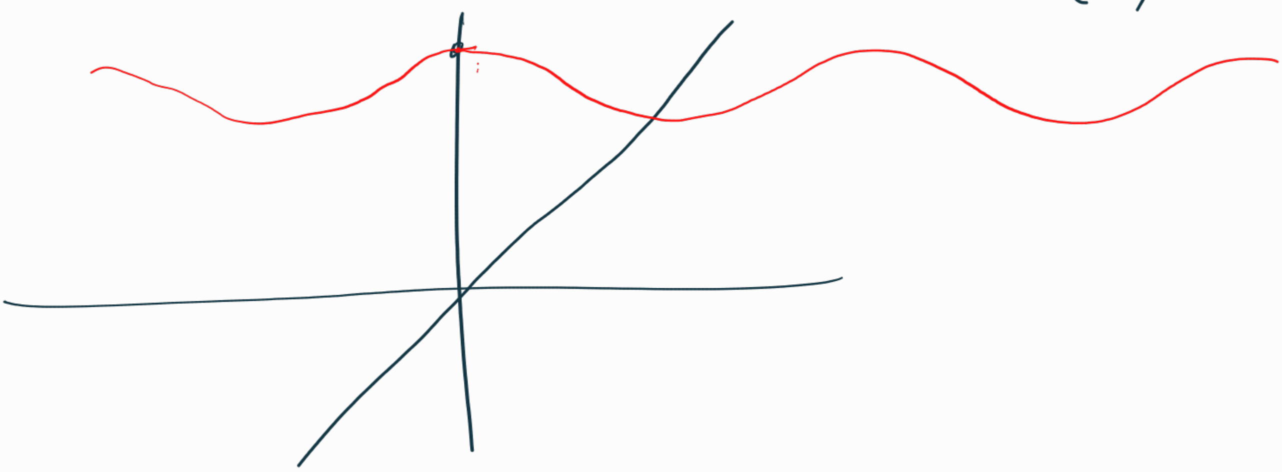
that is, $\bar{x}$ is a fixed point of F

<u>Proof : next time</u>.

<u>Applications</u>

$X = \mathbb{R}$ $\qquad F(x) = 5 + \cos\left(\frac{x}{2}\right)$

$\Rightarrow \exists!\ \bar{x} \in \mathbb{R}$ t.c. $\bar{x} = 5 + \cos\left(\frac{\bar{x}}{2}\right)$.



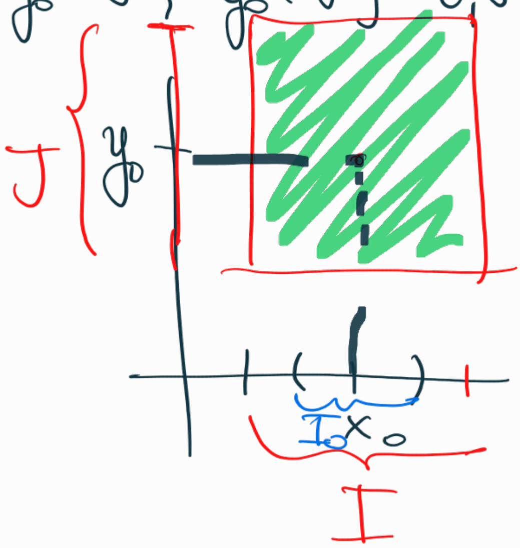
<u>Application</u>: We want to use the contraction theorem in a Functional Analysis setting to prove the following result of local existence and uniqueness of a Cauchy problem for a differential equation.

<u>Theorem</u>

$$(P) \qquad \begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$$

$f : I \times J \longrightarrow \mathbb{R}$ of class $C^1(I \times J)$.

$I = [x_0 - \delta, x_0 + \delta]$, $J = [y_0 - \sigma, y_0 + \sigma]$ $\delta, \sigma > 0$



Then $\exists$ $I_0 = [x_0 - r, x_0 + r]$ and

$\exists !$ $y(x) : I_0 \longrightarrow J$ of class C^1

which is solution of (P)

<u>Proof</u>.

1) I want to reduce the pb. to an integral problem.

I consider the two statements

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases} \qquad y \in C^1(I_0; J) \qquad (P)$$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \quad \forall x \in I_0 \qquad y(x) \in C^0(I_0; J)$$

$$(Q)$$

I claim that (P) $\Longleftrightarrow$ (Q)

Suppose (P) holds. By the fundamental theorem of integral calculus

$$y(x) = \underset{\underset{y_0}{\shortparallel}}{y(x_0)} + \int_{x_0}^{x} \underset{\underset{f(t,y(t))}{\shortparallel}}{y'(t)}\, dt = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

So (P) $\Longrightarrow$ (Q)

Suppose (Q) holds.

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \qquad y(x) \in C^0(I_0; J)$$

$$y(x_0) = y_0$$

Differentiating (Fundam. theorem)

$$y'(x) = f(x, y(x)) \quad \text{and} \quad y(x) \in C^1(I_0; J)$$

---

2) We want to solve (Q).

$$X \doteq C^0(I_0; J) = \{ f \in C(I_0) : |f(x) - y_0| \leq \sigma \}$$

This is a complete metric space with the distance

$$d(f, g) = \max_{I_0} |f(x) - g(x)|.$$

Let $y(x) \in X$. We define an application $T : X \to X$

$$(T(y))(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

So I want to find a $y(x) \in X$ s.t.

$$\boxed{T(y) = y}$$

So, I want to find $r > 0$ s.t. $\nearrow$ defines $I_0 = [x_0 - r, x_0 + r]$

1) $T: X \longrightarrow X$

2) T is a contraction

$\Longrightarrow \exists!\, y$ s.t. $T(y) = y$

i.e. y is a sol^n of (Q)

proof of 1)

of course $(T(y))(x)$ is a C^1 function. I must show that it has values in $J = [y_0 - \sigma, y_0 + \sigma]$, that is,

$$|T(y)(x) - y_0| \leq \sigma \qquad \forall x \in I_0.$$

$$\overset{\shortparallel}{\left| \cancel{y_0} + \int_{x_0}^{x} f(t, y(t))\,dt - \cancel{y_0} \right|}$$

$$\overset{\shortparallel}{\left| \int_{x_0}^{x} f(t, y(t))\,dt \right|} \leq \left| \int_{x_0}^{x} \underbrace{|f(t, y(t))|}_{\substack{\wedge \\ M}}\,dt \right| \leq$$

M by Weierstrass thm

$$\leq M \underbrace{|x - x_0|}_{\substack{\wedge \\ r}} \leq Mr \leq \sigma$$

if we choose $\boxed{r \leq \frac{\sigma}{M}}$

We will finish the proof next time....